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PLANE AND SPHERICAL
TRIGONOMETRY

NAVIGATION AND NATURAL ASTRONOMY

TRIGONOMETRY AND TRIGONOMETRICAL TABLES

BY EDMUND SENKLER

BY T. S. DAVIES AND J. H. PEARSON

LONDON

WASINGTON, D. C. BY THE NATIONAL ACADEMY OF SCIENCES

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ELEMENTS

PLANE AND SPHERICAL

TRIGONOMETRY,

WITH

ITS APPLICATIONS TO THE PRINCIPLES

OF

NAVIGATION AND NAUTICAL ASTRONOMY;

WITH THE

LOGARITHMIC AND TRIGONOMETRICAL TABLES.

BY J. R. YOUNG,

AUTHOR OF "THE ELEMENTS OF ANALYTICAL GEOMETRY;" "ELEMENTS OF
THE DIFFERENTIAL AND INTEGRAL CALCULUS," &c.

TO WHICH IS ADDED

SOME ORIGINAL RESEARCHES IN

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BY T. S. DAVIES, F.R.S.E., F.R.A.S., &c.

LONDON:

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PREFACE.

It is the design of this treatise to establish the theory of Plane and Spherical Trigonometry analytically, and to present that theory, together with some of its most interesting and valuable applications, in a form fitted for elementary instruction.

Of late years several analytical works on Trigonometry have been published in this country; but, as they are confined almost entirely to the theory of the subject, it may be questioned whether, to many young students, they prove much else than so many collections of mere algebraical exercises. Yet a book upon so practical a subject as Trigonometry ought undoubtedly to be something more than this, and ought not to be considered as complete when the various calculations which the science involves, and which its name implies, are wholly omitted.

The symbolical expression of a practical rule, in algebraic language, will often, to the young student, but indistinctly point out the numerical operation. Those much occupied

in mathematical instruction know full well that a learner may readily yield his assent to every step of an algebraic process, be fully satisfied as to the truth of the result to which it leads, may even clearly see a valuable truth involved in it, and may yet be very far from perceiving how to turn it to account in any case of actual calculation. Indeed, algebraical formulas, transform them as we will, cannot always be made to indicate the best mode of arithmetical arrangement; and yet much, as regards facility of operation, depends upon this arrangement in many parts of practical mathematics, but especially in Trigonometry.

In the present volume, therefore, both the theory and the practice of the science have been introduced, every practical formula being illustrated by examples of the numerical calculation, arranged in the proper form. This plan of combining practice with theory, in works like the present, was always adopted by the earlier English writers, and it is to be regretted that recent authors have, in their admiration of foreign methods, departed so widely, in this respect, from the example of their predecessors, dwelling so much as they do upon the symbols, and so little upon the things signified.

In addition to the practical illustration of formulas, a distinct part of the work is devoted to the principles of Navigation and Nautical Astronomy, in which will be found a very short and convenient method of clearing the Lunar Distance, for the purpose of ascertaining the Longitude at

Sea. This method is probably new, although, as the analytical expression for it occurs during the investigation of the well known formula of *Borda*, it is equally probable that it has been noticed before.

The supplement appended to the treatise is from the pen of my valued and accomplished friend, *T. S. Davies, Esq.* Fellow of the Royal Society of Edinburgh, and of the Royal Astronomical Society of London. It will be found to contain several new and interesting researches, which cannot fail to prove acceptable both to the inquiring student and to the more advanced analyst.

J. R. YOUNG.

January 1, 1833.

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PART I.

ELEMENTS OF PLANE TRIGONOMETRY.

CHAPTER I.

EXPLANATION OF THE TRIGONOMETRICAL LINES.

(Article 1.) PLANE TRIGONOMETRY is that branch of pure mathematics of which the primary object is to determine the several parts of a plane triangle from having certain other dependent parts given.

By the *parts* of a plane triangle we mean these six things, viz. the three sides and the three angles, and if any three of these six be given, provided only that a side be among them, the other three may always be determined either by geometrical construction, as shown in the Elements of Geometry, or by numerical computation, as will be seen hereafter.

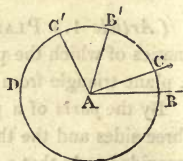
From the foregoing definition it appears that quantities of two kinds, perfectly distinct from each other and admitting of no comparison, are concerned in Trigonometry, viz. *straight lines* and *angles*.

By means of certain happy contrivances, however, the whole business of trigonometry, and, indeed, the general theory of angular magnitude is conducted by help of linear quantities only; the angles themselves not entering into the computations, but certain straight lines dependent upon them and serving as indexes to them.

(2.) Before we explain the nature of these trigonometrical lines, it will be necessary first to show how angular magnitude is measured.

In order to this we must remark that when straight lines are submitted to calculation, all those which are concerned in the same inquiry must be measured in reference to one common standard of measure, called the *linear unit*; the choice of which unit is, however, arbitrary. Thus if we estimate any one of the lines concerned in any inquiry in feet, all must be estimated in feet, and the linear unit adopted will be a *foot*, which is represented by the *numeral unit* 1. Also if one of the lines is measured in yards all must be measured in yards, the linear unit being then a *yard*, which, as before, is represented in the calculation by the numeral unit 1. As far as the accurate representation of the lines are concerned, it is obviously a matter of indifference what length be assumed for the linear unit, for the length of any line will always be expressed numerically by that number which denotes the units it contains, but, for the purpose of facilitating computation, some scales of measure are often preferable to others.

(3) Let now BAC be any angle concerned in any inquiry: then having chosen the linear unit AB , describe the circumference BCD about the centre A . The arc BC may be taken for the measure or representative of the angular magnitude CAB : for let there be any other angle $B'AC'$ about the same centre A ; then we know, by Geometry, that the angle BAC is to the angle $B'AC'$ as the intercepted arc BC to the intercepted arc $B'C'$, (Geometry, p. 102); hence, as the intercepted arcs always vary as the angles, the former may, obviously, be taken to represent the latter.



It is usual to consider the circumference of every circle to consist of 360 equal parts, called *degrees* of that circle; an arc consisting of any number of these, 24 for instance, is called an arc of 24 degrees, and represented for brevity thus, 24° ; moreover each degree is supposed to consist of 60 equal parts, called *minutes*, and each minute of 60 equal parts called *seconds*. To express any number of minutes, we mark one accent over the number, and to express seconds we mark two; thus, $24^\circ 16' 26''$, is 24 degrees 16 minutes 26 seconds. What we say of circular arcs applies equally to the angles which they measure, so that

we call that an angle of 20° whose sides include an arc of 20° or the eighteenth part of a whole circumference.

Let us now speak of the trigonometrical lines before adverted to, and which are introduced for the purpose of reducing the entire theory of angular magnitude to the investigation of linear quantities only; we must, first, however, mention one or two further particulars respecting the arcs to which these lines refer.

(4.) The arc CD which must be *added* to BC to make up a quadrant, or 90° is called the *complement* of the arc BC; and every arc will have a complement, even those which are themselves greater than 90° , provided we consider the arcs measured in the direction BCD &c. as positive, and those measured in the opposite direction as negative; thus the complement CD of the arc BC commences at C where BC terminates, and may be considered as generated by the motion of C, the extremity of the radius AC, in the direction CD; but the complement C_1D of the arc BC_1 , commencing in like manner at the extremity C_1 of the proposed arc, must be generated by the motion of C_1 in the opposite direction, and the angular magnitude BAC_1 , will here be diminished by the motion of AC_1 , in generating the complement; the complement of BAC_1 , or of the arc BC_1 , is, therefore, with propriety considered as negative. Calling the arc BC, or BC_1 , ω , the complement will be $90^\circ - \omega$; thus the complement of $24^\circ 16' 4''$ is $65^\circ 43' 56''$, and the complement of $120^\circ 36' 10''$ is $-30^\circ 36' 10''$.



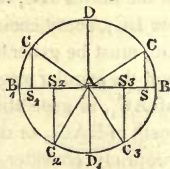
The arc CB_1 , which must be added to BC to make up a semicircle, or 180° , is called the *supplement* of the arc BC. If the arc is greater than 180° , as the arc BC_2 , its supplement C_2B_1 measured in the reverse direction is negative. The expression for the supplement of any arc or angle ω is, therefore, $180^\circ - \omega$; thus the supplement of $110^\circ 30' 20''$ is $69^\circ 29' 40''$, and the supplement of $200^\circ 25'$ is $-20^\circ 25'$.

In the same manner as the complementary and supplementary arcs are considered as positive or negative, according to the direction in which they are measured, so are the arcs themselves positive or negative; thus, still taking B for the commencement of the arcs, as BC is positive BC_3 will be negative. In the doctrine of triangles we consider only positive angles or arcs, and the magnitudes of these are comprised

between $\omega = 0$ and $\omega = 180^\circ$; but in the general theory of angular quantity, we consider both positive and negative angles, according as they are situated above or below the fixed line AB from which they are measured, as the angles CAB, C_3 AB; moreover, an angle may consist of any number of degrees whatever, thus if the revolving line AC set out from the fixed line AB and make n revolutions, and a part the angular magnitude generated is measured by n times 360° , plus the degrees in the additional part.

Of the Sine.

(5.) The sine of an arc or of the angle which it measures, is the perpendicular, from one extremity of the arc, upon the diameter passing through the other extremity: thus CS is the sine of the arc BC; $C_1 S_1$ is the sine of the arc BC_1 ; $C_2 S_2$ is the sine of the arc BC_2 ; $C_3 S_3$ the sine of the arc BC_3 , &c. If the proposed arc were a quadrant, or 90° , the sine DA would be equal to the radius, and, therefore, its numerical value would be 1; the same would be the case if the arc consisted of 3 quadrants, or 270° , or indeed of any odd number of quadrants; for all other arcs the numerical value of the sine will be a proper fraction or decimal. These, it must be observed, are the trigonometrical values of the sines, which are estimated according to the scale $AB = 1$; but it should be remarked that when we know the value of the sine of an arc agreeably to this scale, its value agreeably to any other scale is at once obtained by proportion; thus let R be any value assumed for the radius, and let us write the sine corresponding in capitals, SINE; then $1 : \text{sine} :: R : \text{SINE} = R \times \text{sine}$, so that the sine of an arc, corresponding to any assumed radius, is found by multiplying its trigonometrical sine by that radius; and, on the contrary, the sine according to any value of the radius being known, the trigonometrical sine is found by dividing it by that radius; the number, in fact, which expresses the trigonometrical sine being the ratio of the geometrical line itself to the

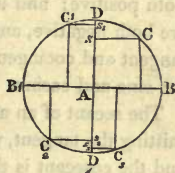


radius, whatever this may be. What we have said of the sine will be easily seen to apply to the other trigonometrical lines. As with the arcs so with the sines, those which lie in opposite directions take opposite signs, those above the fixed line $B_1 B$ being regarded as positive, and those below as negative, so that the sines in the first and second quadrants are positive, those in the third and fourth negative, while in the fifth and sixth they are again positive, and so on.

Every arc or angle has the same sine as its supplement; thus if $B_1 C_1$ is equal to BC it is obvious that BC_1 will be the supplement of BC , and the sine CS of the latter must be equal to the sine $C_1 S_1$ of the former.

Of the Cosine.

(6.) The cosine of an arc or angle is the sine of its complement: thus the cosine of the arc BC is the line Cs , which is, obviously, the sine of the arc DC , the complement of BC . As the several sines are arranged on opposite sides of the diameter $B_1 B$, so the cosines are arranged on opposite sides of the diameter DD_1 ; those on the right of DD_1 being regarded as positive, and those opposite as negative; hence in the first quadrant, the cosines are positive, in the second negative, in the third negative, in the fourth positive, and so on; the cosine of an arc is equal to the cosine of its supplement, but has a different sign.



When the arc is 0 the sine is 0, but the cosine BA is 1; when the arc is 90° , the sine DA is 1, but the cosine is 0; when the arc is 180° the sine is 0, but the cosine is $B_1 A = -1$; when the arc is 270° the sine $D_1 A$ is -1 , but the cosine is 0; and when the arc is 360° the sine is 0, and the cosine 1, as at first, and so on.

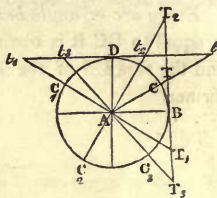
It is plain that the cosine of an arc is always equal to that part of the radius which is intercepted between the sine of that arc and the centre. Thus referring to the figure in (5) AS is equal to the cosine of BC , and AS_2 to the cosine of $BC_1 C_2$, or of $BC_3 C_2$.

Of the Tangent, Cotangent, Secant, and Cosecant.

(7.) The tangent of an arc, and, therefore, of the angle which it measures, is a line drawn from one extremity of the arc, touching it at that extremity, and terminating in the diameter produced, drawn through the other extremity: thus BT is the tangent of the arc BC .

The cotangent is the tangent of the complement: thus Dt is the cotangent of the arc BC .

It is easy to trace the changes which these two lines undergo as the arc BC increases from 0, for which value the tangent is obviously 0, and the cotangent infinite. Observing the same rules here as for the sine and cosine, we see that in the first quadrant the tangent and cotangent



are both positive, in the second the tangent BT_1 and cotangent Dt_1 are both negative; in the third the tangent BT_2 and cotangent Dt_2 are both positive; and in the fourth the tangent BT_3 and cotangent Dt_3 are both negative, and so on; but as we shall soon see, the signs of the tangent and cotangent may always be at once inferred from those of the sine and cosine.

The secant of an arc is that portion of the prolonged diameter, limiting the tangent, which is included between the centre and tangent; and the cosecant is the secant of the complement. Thus in the last figure AT is the secant of the arc BC , and At the cosecant.

In the four trigonometrical lines, sine, cosine, tangent, and cotangent, we have seen that each is posited in one or other of two directly opposite directions, and that, therefore, one or other of the opposite signs $+$ and $-$, prefixed to the numerical value of any such line, served to point out the proper direction for any particular value of the arc or angle. But as the secant and cosecant continually vary in direction, as well as in magnitude with the arc or angle, the geometrical position of either of these lines does not so clearly indicate to us the sign with which it should be represented. The proper sign, however, is always readily ascertained from knowing the signs of the sine and cosine, for upon these two lines all the others depend, as we shall shortly show.

(8.) Besides the six trigonometrical lines now defined there are three others, sometimes, although but seldom, employed; these are the *versed sine* or *sagitta*, the *covered sine*, and the *suversed sine*. The versed sine of an arc BC (see fig. to art. 5) is the line BS between the commencement of the arc and the sine; it is always equal to the radius minus, the cosine, and, therefore, is always positive. The covered sine is the versed sine of the complement, so that the covered sine of BC is Ds (see fig. to art. 6); also the suversed sine is the versed sine of the supplement. As the versed sine of any arc must be positive, it follows that the covered sine and suversed sine must always be positive.

(9.) The following is the way in which the trigonometrical lines, connected with any arc or angle ω , are expressed in computation;

The sine of ω is expressed thus, $\sin. \omega$

cosine of ω	.	.	$\cos. \omega$
tangent of ω	.	.	$\tan. \omega$
cotangent of ω	.	.	$\cot. \omega$
secant of ω	.	.	$\sec. \omega$
cosecant of ω	.	.	$\csc. \omega$
versed sine of ω	.	.	$\text{vers. } \omega$
covered sine of ω	.	.	$\text{covers. } \omega$
suversed sine of ω	.	.	$\text{suvers. } \omega$

From knowing the numerical value of any one of these lines, those of all the others may be obtained; thus, let the sine be given, then since the radius sine and cosine always form a right-angled triangle, of which the hypotenuse is the radius = 1, (see the fig. in art. 5), we have

$$\cos. \omega = \sqrt{1 - \sin.^2 \omega}.$$

Again, since the triangle formed by the radius, sine, and cosine, is always similar to that formed by the secant, tangent, and radius, and to that formed by the cosecant, radius, and cotangent, as the student will at once see by sketching these lines for any arc, it follows, from the proportionality of

the sides of similar triangles, that

$$\begin{aligned}\tan. \omega &= \frac{\sin. \omega}{\cos. \omega}, & \cot. \omega &= \frac{\cos. \omega}{\sin. \omega} = \frac{1}{\tan. \omega} \\ \sec. \omega &= \frac{1}{\cos. \omega}, & \operatorname{cosec.} \omega &= \frac{1}{\sin. \omega} = \sqrt{1 + \cot.^2 \omega};\end{aligned}$$

and, from these expressions, we at once see that the signs of the several lines, as well as their numerical values, are deducible from those of the sine and cosine.

Now the numerical expression for $\sin. \omega$, for all values of ω , from $\omega = 0$ to $\omega = 90^\circ$, (between which limits every possible value is comprised) are actually computed by methods to be hereafter explained, and thence the values of the other trigonometrical lines are deduced. These values are then arranged as in table III, at the end, and form a *table of natural sines, cosines, &c.* By help of such a table we may readily find the values of the same lines, computed to any other radius R ; for as observed at (5) we shall merely have to multiply the tabular value by R . Writing; therefore, for distinction sake, the words $\sin.$, $\cos.$, &c. in capitals, when the value of the radius is other than unity, the foregoing equations are the same as

$$\begin{aligned}\frac{\operatorname{TAN.} \omega}{R} &= \frac{\operatorname{SIN.} \omega}{\operatorname{COS.} \omega}, & \frac{\operatorname{COT.} \omega}{R} &= \frac{R}{\operatorname{TAN.} \omega} \\ \frac{\operatorname{SEC.} \omega}{R} &= \frac{R}{\operatorname{COS.} \omega}, & \frac{\operatorname{COSEC} \omega}{R} &= \frac{R}{\operatorname{SIN.} \omega};\end{aligned}$$

and thus by substituting in any trigonometrical formula $\frac{\operatorname{SIN.} \omega}{R}, \frac{\operatorname{COS.} \omega}{R}$

&c. for $\sin. \omega, \cos. \omega$, &c. the formula will become generalized so as to hold good for any value of the radius whatever.

(10.) It is obvious that when any trigonometrical formula is thus generalized every term in it will be the same abstract number as in the original formula; whatever powers or roots of the lines enter the formula they will always be divided by the same powers or roots of the radius R . The denominators will all be removed by multiplying each term by the highest power of R , which enters, and the result will ne-

cessarily be a homogeneous expression; that is, every term will have the same dimensions, or will involve as factors the same number of lines. Hence, in order to generalize any trigonometrical formula, or to render it independent of any particular value of R , it will be necessary merely to introduce into the several terms such powers of R as will render them all of the same dimension. For example, the following formula, viz.

$$\sin. (A + B) = \sin. A \cos. B + \sin. B \cos. A;$$

in which the term on the left is of one dimension, and the terms on the right are each of two dimensions, will become homogeneous by introducing the factor R into the left hand number, so that when this is the value of the radius instead of unity, the formula will be

$$R \sin. (A + B) = \sin. A \cos. B + \sin. B \cos. A;$$

each term being the product of two lines.

In like manner the formula

$$\cos. 4 A = 8 \cos.^4 A - 8 \cos.^2 A + 1,$$

becomes when the radius is R instead of unity

$$R^3 \cos. 4 A = 8 \cos.^4 A - 8 R^2 \cos.^2 A + R^4;$$

the powers of R being introduced so as to render each term of four dimensions.

From the preceding definitions and remarks the following simple properties are immediately deducible, viz.

1. The sine of an arc is equal to half the chord of twice that arc.
2. The chord of 60° being equal to the radius (Geom. p. 119), therefore, the sine of 30° , or the cosine of 60° , is equal to half the radius.¹
3. Hence, from the expression for the secant at the top of the preceding page, the secant of 60° is equal to the diameter of the circle.
4. The tangent of 45° is equal to the cotangent, and, therefore, to the radius, (see fig. to art. 7.)

(11). We shall terminate this introductory chapter with a table exhibiting the correlative values of the trigonometrical lines, situated in different quadrants; it is readily constructed from the values of the sine and cosine, by help of the relations in (9), bearing in mind that an arc and its supplement have the same sine.

Table of the Correlative Values of the Trigonometrical Lines.

arc.	sin.	cos.	tan.	cot.	sec.	cosec.
0°	0	1	0	∞	1	∞
ω	$+\sin. \omega$	$+\cos. \omega$	$+\tan. \omega$	$+\cot. \omega$	$+\sec. \omega$	$+\operatorname{cosec.} \omega$
90°	1	0	∞	0	∞	1
$90^\circ + \omega$	$+\cos. \omega$	$-\sin. \omega$	$-\cot. \omega$	$-\tan. \omega$	$-\operatorname{cosec.} \omega$	$+\sec. \omega$
180°	0	-1	0	$-\infty$	-1	$-\infty$
$180^\circ + \omega$	$-\sin. \omega$	$-\cos. \omega$	$+\tan. \omega$	$+\cot. \omega$	$-\sec. \omega$	$-\operatorname{cosec.} \omega$
270°	-1	0	$-\infty$	0	$-\infty$	1
$270^\circ + \omega$	$-\cos. \omega$	$+\sin. \omega$	$-\cot. \omega$	$-\tan. \omega$	$+\operatorname{cosec.} \omega$	$-\sec. \omega$
360°	0	1	0	∞	1	∞

This last line is the same as the first; and any line will, obviously, remain unaltered if we add to the corresponding arc a whole circumference or any number of circumferences. If we take ω negatively, we may extend the table as follows:

$-\omega$	$-\sin. \omega$	$+\cos. \omega$	$-\tan. \omega$	$-\cot. \omega$	$+\sec. \omega$	$-\operatorname{cosec.} \omega$
$90^\circ - \omega$	$+\cos. \omega$	$+\sin. \omega$	$+\cot. \omega$	$+\tan. \omega$	$+\operatorname{cosec.} \omega$	$+\sec. \omega$
$180^\circ - \omega$	$+\sin. \omega$	$-\cos. \omega$	$-\tan. \omega$	$-\cot. \omega$	$-\sec. \omega$	$+\operatorname{cosec.} \omega$
$270^\circ - \omega$	$-\cos. \omega$	$-\sin. \omega$	$+\cot. \omega$	$+\tan. \omega$	$-\operatorname{cosec.} \omega$	$-\sec. \omega$
$360^\circ - \omega$	$-\sin. \omega$	$+\cos. \omega$	$-\tan. \omega$	$-\cot. \omega$	$+\sec. \omega$	$-\operatorname{cosec.} \omega$

and by continuing this series of arcs the same values of the trigonometrical lines would obviously recur as before.

It is obvious that the cosine of a negative arc, whether less or greater than a quadrant, is the same as the cosine of the same arc, taken positively; but the sine of a negative arc, although the same in magnitude as that of an equal positive arc, has an opposite sign: hence, by the equations at (9), the sine, tangent, cotangent, and cosecant, will have opposite signs to those of the same arc taken positively; but the cosine and secant will have the same signs.

CHAPTER II.

FORMULAS AND RULES FOR THE SOLUTION OF PLANE TRIANGLES.

(12.) We shall now proceed to investigate rules for the solution of all the cases of plane triangles.

Right-angled triangles.

As right-angled triangles are those whose several parts are the most easily determined we shall consider them first.

Let ABC be any right-angled plane triangle, and with AB as a radius describe the arc Ba. If AB were unity BC would be the tangent, and AC the secant of the angle A; as it is, however, these lines are equal to AB times the trigonometrical tangent and secant (5), that is,

$$BC = AB \tan. A, \quad AC = AB \sec. A.$$

Also, by taking the hypotenuse for the radius, we have

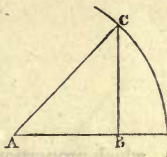
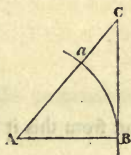
$$BC = AC \sin. A, \quad AB = AC \cos. A.$$

These four equations, together with the geometrical property

$$AC^2 = AB^2 + BC^2,$$

enable us to solve every case of right-angled triangles.

(13.) In applying these formulas, it must be remembered that the trigonometrical lines which they involve are according to the scale of radius = 1; they are computed and registered in the tables of *natural sines and tangents*. The tables of *logarithmic sines and tangents* are



not, however, computed to this radius, on account of the inconvenience which would attend the continual use of negative indices in all the sines and cosines; but they are computed to a radius of 10^{10} . Hence, in all formulas of trigonometry intended for logarithmic computation, the radius R must always be introduced, so as to make the terms homogeneous; and, although in the formulas which will be hereafter given, we shall but seldom encumber the expressions by actually inserting in them R , and its powers, yet the computist must not fail to take account of them in the logarithmic process.

Introducing R into the foregoing equations we may write them thus:

$$\frac{R}{AB} = \frac{\tan. A}{BC}, \quad \frac{R}{AB} = \frac{\sec. A}{AC}, \quad \frac{R}{AC} = \frac{\sin. A}{BC}, \quad \frac{R}{AC} = \frac{\cos. A}{AB};$$

and all these equations may be comprehended in a single rule, expressed as below

As the tabular radius

: the radius in the figure,

: : any tabular line

: the corresponding line in the figure;

and from this it immediately follows that

Any tabular line

: corresponding line in the figure,

: : any other tabular line

: corresponding line in the figure;

which proportion, obviously, comprehends the former.

It appears from this rule that when we want to find a side, we must begin the proportion with a given tabular line, that is, either with the tabular radius, of which the logarithm is 10, or else with the tabular sine, cosine, &c. of a given angle; but when we want to find an angle, then we must invert this proportion, beginning with a given side, which must be made the geometrical radius, as no other tabular line but the radius will be given, seeing that angles are in this case unknown.

(14.) In operating with logarithms, the logarithm of the first term of

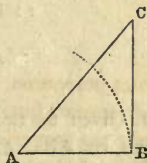
the proportion must be subtracted from the sum of the logs. of the other two, to obtain the logarithm of the sought fourth term; and thus the logarithmic process will consist of five lines or rows of figures. If, however, the first term, or that to be subtracted were 10, we might save a line, by adding the two other logs. together, and rejecting 10 in the index; when the first term is not 10 we may still save a line by the following artifice, viz. instead of putting down for the first term the log. given by the table, put down its deficiency from the number 10, which may be done with as much readiness as transcribing the number itself, provided we begin at the left-hand figure and subtract each in succession from 9, till we come to the last significant figure, which must be taken from 10; we shall thus have instead of the logarithm, what is called its *arithmetical complement*, which, being added in with the other two terms, rejecting 10 from the index, must give the same result as if we had subtracted the log. of the first term from the sum of the other two. An example or two will fully illustrate what has now been said.

EXAMPLES.

(15.) 1. Given the angles and the base to find the perpendicular and hypotenuse, viz. $A = 53^{\circ} 8'$, $AB = 288$.

1. To find the Perpendicular BC.

As a side is here required, we must begin with a tabular line; we shall choose for simplicity the tabular radius, to save a reference to the table, as we know the log. of this to be 10. Taking then the known line AB for radius in the figure, we have



Rad.	.	.	10
: AB	288		2.4593925
:: tan. A	53° 8'		10.1249898
: BC	384.05		2.5843823;

BC might have been found by making any other side radius, although not quite so easily, as we should then have had to seek out in the table the tabular line for the first term, corresponding to the known line AB; thus if AC had been made radius, then the tabular line we should have commenced with, would have been that corresponding to AB, viz. the cosine of the angle A. If CB had been made radius we should have commenced with the cotangent of A, that is, the tangent of C, for such would be the tabular line corresponding to BA.

II. To find the Hypotenuse AC.

Preserving the same radius we have,

Rad.	.	10
: AB	288	2·4593925
:: sec. A	53° 8'	10·2218814
: AC	480·036	2·6812739.

If we had made AC radius, the proportion would have been $\cos. A : AB :: \text{rad.} : AC$. By way of showing the use of the arithmetical complement, let us determine AC by this proportion

cos. A	53° 8' arith. comp.	0·2218814
: AB	288	2·4593925
:: Rad.	.	10
: AC	480·036	2·6812739.

2. Given the two perpendicular sides to find the hypotenuse and angles, viz. $AB = 472$, $BC = 765$, (see last fig.)

I. To find the Angle A.

We must here, agreeably to the rule, begin with a given side, say AB, which we shall make radius.

AB	472 arith. comp.	7.3260580
: Rad.	.	10
:: BC	765	<u>2.8836614</u>
: tan. A	$58^{\circ} 19' 32''$	<u>10.2097194.</u>

II. To find the Hypotenuse.

Here we must begin with a tabular line; we shall choose the radius.

Rad.	.	10
: AB	472	2.6739420
:: sec. A	$58^{\circ} 19' 32''$	<u>10.2797645</u>
: AC	898.89	<u>2.9537065.</u>

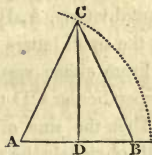
Or without employing the angle A we may determine AC by the formula.

$$AC = \sqrt{AB^2 + BC^2}.$$

3. Given two sides and the included angle of an isosceles triangle ABC to find the other parts.

$$AC = BC = 288, \angle ACB = 78^{\circ} 12'.$$

Let the perpendicular CD be drawn, then since it will bisect the angle C, we shall have given in the right-angled triangle



* For the method of determining the angle corresponding to any tabular number to seconds, see the introductory explanation prefixed to the tables.

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ADC, $AC = 288$, $ACD = 30^\circ 6'$ $\therefore A = 90^\circ - 39^\circ 6' = 50^\circ 54'$;

hence, to find AD, we have by making AC radius,

Rad.	.	10
: AC	288	2.4593925
: : cos. A	$50^\circ 54'$	9.7998062
: AD	181.635	<u>2.2591987</u>

$$\therefore AB = 363.270.$$

4. Given the base $AB = 53.42$, and the perpendicular $BC = 75.18$, to find the hypotenuse and angles?

$$A = 54^\circ 36' 14'', C = 35^\circ 23' 46'', AC = 72.23.$$

5. Given the hypotenuse $AC = 643.7$, and the base $AB = 473.8$, to find the other parts?

$$A = 42^\circ 36' 12'', C = 47^\circ 23' 48'', BC = 35.87.$$

6. Given the angle $A = 37^\circ 2' 43''$, and the hypotenuse $AC = 173.2$ to find the other parts?

$$C = 52^\circ 57' 17'', AB = 138.24, BC = 104.34.$$

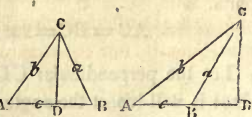
(16.) We shall now proceed to investigate rules and formulæ for the solution of triangles in general.

Oblique-Angled Triangles.

Let ABC be any plane triangle, and let us denote the angles by the capital letters, A, B, C, at their vertices, and the sides opposite to them by the small letters a, b, c .

From either vertex, as C, draw the perpendicular CD to the opposite side.

Then the sine of A to the radius b will, obviously, be the line CD, and the value of this sine in terms of the trigonometrical sine of the same angle to radius 1 is (art. 5,) $CD = b \sin. A$. In like manner the sine of B to the radius a , is the same line CD, whose value, therefore, in term of the trigonometrical sine, is $CD = a \sin. B$; consequently, by



equating these two values of CD, we have

$$a \sin. B = b \sin. A \therefore \frac{a}{b} = \frac{\sin. A}{\sin. B}.$$

This equation immediately furnishes us with an important rule, which may be expressed as follows.

Any side of a triangle is to any other side as the sine of the angle, opposite to the former, is to the sine of the angle opposite to the latter.

Whenever, therefore, we know two sides and an angle opposite to one of them, or two angles and a side opposite to one of them, the other three parts of the triangle may always be determined by help of this rule.

The cosine of A, to the radius b , is the line AD; and, therefore, AD, in terms of the trigonometrical cosine of A, is $AD = b \cos. A$. In like manner the cosine of B to the radius BC, is BD, which, in terms of the trigonometrical cosine, is $BD = a \cos. B$; if the angle B is obtuse, as in the second of the above diagrams, $\cos. B$ will be negative; hence whether it be acute or obtuse we shall have for the side AB the expression

$$c = a \cos. B + b \cos. A;$$

in which the proper signs of the cosines are supposed to be involved in their expressions.

If instead of drawing the perpendicular from C we had drawn it from B, it is easy to see the result we should have obtained; for then considering B the vertical angle instead of C, or supposing the triangle to be turned about till B actually becomes the vertical angle, then commencing at the vertex, the arrangement of the angles will now be B, C, A; these, therefore, should respectively be substituted for C, A, B, in the above formula; also the arrangement of the sides will be a, b, c , instead of b, c, a , as at first, so that these letters must be replaced by the former: consequently, our equation will become

$$b = c \cos. A + a \cos. C.$$

If, on the contrary, A be made the vertical angle, then the order of the angles will be A, B, C, and of the sides c, a, b , and these must supply the places, of C, A, B, and b, c, a , in the first formula, so that we shall

then have

$$a = b \cos. C + c \cos. B.$$

Collecting these equations together we have,

$$\left. \begin{aligned} a &= b \cos. C + c \cos. B \\ b &= c \cos. A + a \cos. C \\ c &= a \cos. B + b \cos. A \end{aligned} \right\} \dots (1);$$

and these equations contain the whole theory of plane trigonometry. They involve all the six parts of a triangle, the three angles, and the three sides; and, as the equations are three in number, any three of the parts, considered as unknown quantities, may be determined, provided only the other three are known; but fewer than three being given will not be sufficient to determine the others, as then there would be a greater number of unknowns than of equations.

We must remark too that the three given quantities must not be the three angles simply, because the three other quantities a, b, c , severally enter the three terms of each equation, so that if we were to multiply each equation, by any assumed factor whatever, m , the values resulting from the elimination of A, B, C , would, obviously, be the same for ma, mb, mc , as for a, b, c ; thus, showing that the data are not sufficient to *determine* any triangle, but belong equally to innumerable triangles, all, however, similar to each other.

(17.) It appears then that the solutions to all the cases of plane triangles are derivable from the equations (1), under different hypotheses, as to the three unknown quantities, and we might now with but little trouble proceed to deduce these solutions, one after another, from these equations: thus suppose the three sides a, b, c , were given, then multiplying the first equation by a , the second by b , and the third by c , we have

$$a^2 = ab \cos. C + ac \cos. B$$

$$b^2 = bc \cos. A + ba \cos. C$$

$$c^2 = ac \cos. B + bc \cos. A;$$

and subtracting each of these from the sum of the other two, we get

$$\left. \begin{aligned} b^2 + c^2 - a^2 &= 2bc \cos. A \therefore \cos. A = \frac{b^2 + c^2 - a^2}{2bc} \\ a^2 + c^2 - b^2 &= 2ac \cos. B \therefore \cos. B = \frac{a^2 + c^2 - b^2}{2ac} \\ a^2 + b^2 - c^2 &= 2ab \cos. C \therefore \cos. C = \frac{a^2 + b^2 - c^2}{2ab} \end{aligned} \right\} \dots (2);$$

and thus the values of the cosines of the required angles become known, and by searching in the table of natural sines and cosines we shall find the angles to which they belong.

It is necessary to remark here that in almost every trigonometrical calculation it is advisable to conduct the operation by means of logarithms, in order to avoid lengthy and tiresome multiplications, divisions, and extractions; so that it becomes a matter of consequence to express all our general rules and formulas in a form, adapted as much as possible to logarithmic calculation, that is, the operations indicated by the formulas should be those of multiplication, division, involution, and evolution, and not those of addition and subtraction.

The formulas just deduced for the angles of a triangle, when the sides are given, do not appear in a form adapted to logarithmic computation; and the same would be found to be the case with the various other formulas directly deducible from the general equations (1); nor would it be easy, without the aid of other and independent properties, to convert these expressions into the desired form. Although, therefore, it is true, as we have stated above, that formulas for all the cases of plane trigonometry may be deduced from the equations (1), yet, on account of the inconvenient form these formulas assume, it becomes necessary for us to seek assistance from other sources. Now there exists two general trigonometrical formulas, which may be considered as forming the foundation of the whole theory of angular magnitude, and which, in conjunction with what is laid down above, will enable us to deduce formulas suited to logarithmic calculation for all the cases of plane triangles.

(18.) There are various ways of investigating these formulas; we shall adopt that which appears to us the most simple and general.

It was given by *M. Sarrus* in the *Annales des Mathématiques*, tom. xi.

Given the sines and cosines of two arcs or angles, to find the sine and cosine of their sum and difference.

Let $AM = a$, and $AN = a'$, be any two arcs of the circle, the radius being unity, then drawing the chord of the arc $NM = a - a'$, we shall have from the triangle NMG right angled at G ,

$$MN^2 = NG^2 + MG^2 = (CQ - CP)^2 + (PM - NQ)^2;$$

which may be written thus,

$$\text{chd.}^2(a - a') = (\cos. a' - \cos. a)^2 + (\sin. a - \sin. a')^2.$$

By actually squaring the expressions in the right-hand member of this equation, and recollecting that

$$\sin.^2 a + \cos.^2 a = 1, \quad \sin.^2 a' + \cos.^2 a' = 1,$$

we have

$$\text{chd.}^2(a - a') = 2 - 2 \cos. a \cos. a' - 2 \sin. a \sin. a' \dots (1).$$

Suppose now that $a' = 0$, then we have

$$\text{chd.}^2 a = 2 - 2 \cos. a.*$$

As this expression is true for any arc whatever, it is true for the arc $a - a'$, so that

$$\text{chd.}^2(a - a') = 2 - 2 \cos. (a - a') \dots (2).$$

Comparing together the second members of (1) and (2) we obtain

$$\cos. (a - a') = \cos. a \cos. a' + \sin. a \sin. a' \dots (1).$$

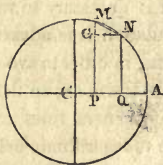
As this is true for all values of a, a' , it is true when $a - a'$ is put for a' , so that

$$\cos. a' = \cos. a \cos. (a - a') + \sin. a \sin. (a - a');$$

in which equation, if we substitute the value of $\cos. (a - a')$ given by (1), we have

$$\cos. a' = \cos. a^2 \cos. a' + \cos. a \sin. a \sin. a' + \sin. a \sin. (a - a');$$

* This property is also proved in the Geometry, p. 92, Scholium.



from which, by putting for $\cos.^2 a$ its value $1 - \sin.^2 a$, we get

$$\sin. (a - a') = \sin. a \cos. a' - \sin. a' \cos. a \dots (II).$$

Lastly, putting $(a + a')$ for a , in the equations (I) and (II), we have,

$$\cos. a = \cos. (a + a') \cos. a' + \sin. (a + a') \sin. a'.$$

$$\sin. a = \sin. (a + a') \cos. a' - \cos. (a + a') \sin. a'.$$

In order to obtain from these equations the expressions for $\sin. (a + a')$ and $\cos. (a + a')$, multiply the first by $\sin. a'$, the second by $\cos. a'$, and add, and we thus get

$$\sin. (a + a') = \sin. a \cos. a' + \sin. a' \cos. a \dots (III).$$

Multiply the first by $\cos. a'$, the second by $\sin. a'$, and subtract, and we get

$$\cos. (a + a') = \cos. a \cos. a' - \sin. a \sin. a' \dots (IV).$$

The four general formulas thus deduced may be written as follows:

$$\left. \begin{aligned} \sin. (a \pm a') &= \sin. a \cos. a' \pm \sin. a' \cos. a \\ \cos. (a \pm a') &= \cos. a \cos. a' \mp \sin. a \sin. a' \end{aligned} \right\} \dots (A).$$

(19.) The first of these immediately furnish the two following, viz.

$$\sin. (a + a') + \sin. (a - a') = 2 \sin. a \cos. a'$$

$$\sin. (a + a') - \sin. (a - a') = 2 \sin. a' \cos. a;$$

from which

$$\frac{\sin. (a + a') + \sin. (a - a')}{\sin. (a + a') - \sin. (a - a')} = \frac{\sin. a}{\cos. a} \cdot \frac{\cos. a'}{\sin. a'} = \frac{\tan. a}{\tan. a'} \text{ (art. 8)}$$

If, therefore, we put

$$a + a' = A, \quad a - a' = B \quad \therefore \quad a = \frac{1}{2} (A + B), \quad a' = \frac{1}{2} (A - B),$$

we shall have

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{1}{2} (A + B)}{\tan. \frac{1}{2} (A - B)}.$$

Now we have already seen that in any plane triangle

$$\sin. A : \sin. B :: a : b$$

$$\therefore \sin. A + \sin. B : \sin. A - \sin. B :: a + b : a - b;$$

consequently, from the equation above,

$$\frac{a + b}{a - b} = \frac{\tan. \frac{1}{2} (A + B)}{\tan. \frac{1}{2} (A - B)};$$

that is to say, *in any plane triangle the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

By help of this rule we may determine the remaining parts of the triangle, when we know two sides a, b , and the included angle C ; for knowing C we know also $\frac{1}{2} (A + B) = \frac{1}{2} (180^\circ - C)$; and $\frac{1}{2} (A - B)$ is determined by this rule; therefore, as the half sum added to the half difference of two quantities gives the greater, and subtracted gives the less; we thence readily obtain the angles A and B , and then the third side c , by (16).

We have thus deduced commodious rules fitted for logarithmic computation, for the solution of the first two cases of plane triangles: it remains to furnish a rule for the third case.

(20.) Referring to the expression for $\cos. A$ at (17), it is plain that since

$$b^2 + c^2 = (b + c)^2 - 2bc,$$

and, therefore,

$$b^2 + c^2 - a^2 = (b + c + a)(b + c - a) - 2bc;$$

that expression may be put under the form

$$\cos. A = \frac{(b + c + a)(b + c - a) - 2bc}{2bc} - 1.$$

Now supposing the arcs a, a' , in equation (A), to be equal to each other, and to $\frac{1}{2} A$, we have from the second of them

$$\cos. A = \cos.^2 \frac{1}{2} A - \sin.^2 \frac{1}{2} A$$

$$1 = \cos.^2 \frac{1}{2} A + \sin.^2 \frac{1}{2} A$$

$$\text{by addition, } \cos. A = 2 \cos.^2 \frac{1}{2} A - 1$$

$$\text{by subtraction, } \cos. A = 1 - 2 \sin.^2 \frac{1}{2} A;$$

by substituting the first of these values in the foregoing equation, and putting for brevity S for the sum of the three sides of the triangle, we have

$$\cos. \frac{1}{2} A = \sqrt{\frac{\frac{1}{2} S (\frac{1}{2} S - a)}{bc}} \dots (1).$$

We can just as readily obtain a second formula by means of the other expression for $\cos. A$; for substituting it in equation (2), art. (17), we have

$$\begin{aligned} 2 \sin.^2 \frac{1}{2} A &= 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - b^2 - c^2 + 2bc}{2bc} = \frac{a^2 - (b - c)^2}{2bc} \\ &= \frac{(a + b - c)(a - b + c)}{2bc}; \end{aligned}$$

consequently,

$$\sin. \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2} S - c)(\frac{1}{2} S - b)}{bc}} \dots (2);$$

and by dividing this expression by the former we get a third formula, viz.

$$\tan. \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2} S - b)(\frac{1}{2} S - c)}{\frac{1}{2} S (\frac{1}{2} S - a)}} \dots (3).$$

(21.) We thus have three distinct formulas for the determination of the angles of a triangle when the three sides are given, and all of them are adapted to logarithmic computation. It is not, however, always a matter of indifference which of these formulas we employ, as in certain cases one may be preferable to another. Thus, if we knew beforehand, or could foresee that the sought angle $\frac{1}{2} A$ would be very nearly equal to 90° , then it would be improper to employ the formula (2), because we should be very likely to commit error in taking out the angle, seeing that for an angle very near 90° the seven first decimals in the sine coincide with those in the sines of several other angles in its vicinity, or which differ each from the proposed angle by only a few seconds.

If the logarithmic tables, which we employ, are calculated to seconds, as the large tables, of *Taylor* or of *Bugay*, then the sought angle when near 90° , may be accurately determined to the nearest second, either from its cosine or from its tangent, as the values of these trigonometrical lines, at this part of the table differ considerably from each other, even when the

arcs are nearly equal. But if the table employed is not calculated to seconds, then the sought angle, when near 90° , should be determined from its cosine, and not from its tangent; because in approaching to 90° the tangents increase by very unequal differences, and, as in proportioning for the seconds, we proceed on the supposition that the tangents increase equally through $60''$, we shall be in danger of committing error in thus determining the seconds. As the cosines decrease more regularly towards the extremity of the quadrant than the tangents increase, it will, therefore, be safest to determine such arcs from their cosines.

When the sought angle is very small it will be best to determine it from its sine; although the tangent may be used with safety.

Solution of Plane Triangles in general.

(22.) We shall now proceed to apply the rules and formulas which we have just investigated to the several cases of plane triangles, repeating the rule at the head of each case.

CASE I.

When a side and its opposite angle are among the given parts.

RULE.

Sine of given angle,

: its opposite side

: : sine of any other angle

: its opposite side.

Also, any given side,

: sine of its opposite angle

: : any other side

: sine of its opposite angle.

As the same sine belongs both to an angle and to its supplement, it may

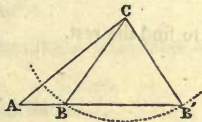
seem doubtful in determining an angle of a triangle from its sine, whether to take the acute angle given by the tables or the obtuse angle which is its supplement.

The following precepts will remove all doubt on this point.

1. If the given angle is obtuse the sought angle must be acute. This is obvious, because a triangle cannot have two obtuse angles.

2. If the given angle be acute, and the side opposite to it greater than the side opposite to the sought angle, this must be acute; for the greater angle must be opposite to the greater side.

3. But when the side opposite to the given angle is less than that opposite to the sought angle, this may be either acute or obtuse, so that two triangles exist under the proposed conditions, and the problem in question admits, therefore, of two solutions. The annexed diagram shows that with two given sides AC , CB , and the acute angle A , opposite to one of them, we may always construct two triangles, ABC , $AB'C$; where the angle B , opposite to the other given side in the one triangle, will be the supplement of the corresponding angle B' in the other, provided CB is less than CA .



EXAMPLES.

(23.) 1. In the triangle ABC are given

$$AB = 137, AC = 153, B = 78^\circ 13',$$

to find the remaining parts.

1. *To find the Angle C.*

As AC	.	153	.	arith. comp.	7.8153086
: $\sin. B$	.	$78^\circ 13'$	.		9.9907502
:: AB	.	137	.		2.1367206
: $\sin. C$	.	$61^\circ 13' 47''$	.		<u>9.9427794.</u>

The obtuse angle, which is the supplement of this, is not admissible, because the side opposite to the given angle is greater than the side opposite the required one.

II. To find the side CB.

The angle A is equal to $180^\circ - (B + C) = 180^\circ - 139^\circ 26' 47'' = 40^\circ 33' 13''$, therefore,

As sin. B	$78^\circ 13'$	.	arith. comp.	0.0092498
: AC	153	.	.	2.1846914
:: sin. A	$40^\circ 33' 13''$	.	.	9.8130198
: CB	101.617	.	.	<u>2.0069610.</u>

2. In the plane triangle ABC are given

$$AC = 216, CB = 117, A = 22^\circ 37',$$

to find the rest.

I. To find the Angle B.

As BC	117	.	arith. comp.	7.9318141
: sin. A	$22^\circ 37'$	.	.	9.5849685
:: AC	216	.	.	2.3344538
: sin. B	$45^\circ 13' 55''$ or $134^\circ 46' 5''$.	.	.	<u>9.8512364</u>

The angle B is, in this example, ambiguous, because the side opposite the given angle is less than that opposite the required one.

II. To find the third side AB.

The angle C is equal to $180^\circ - (A + B) = 112^\circ 9' 5''$, provided we take B acute; therefore,

As sin. A	$22^\circ 37'$	.	arith. comp.	0.4150315
: BC	117	.	.	2.0681859
:: sin. C	$112^\circ 9' 5''$	.	.	9.9667005
: AB	281	.	.	<u>2.4499179.</u>

SCHOLIUM.

In each of the foregoing examples where two sides, and an angle opposite to one, are given, we have found it necessary to find the angle opposite to the other given side before we could apply the rule to the determination of the third side; so that the determination of this third side requires two proportions, and there is no logarithmic method which will lead us to it by a shorter process. It may, however, be deduced directly from the formula at (17), viz.

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc};$$

which gives

$$c = AB = b \cos. A \pm \sqrt{a^2 - b^2 \sin.^2 A};$$

which expression is, however, not adapted to logarithmic computation.

3. In the plane triangle ABC are given

$$A = 44^\circ 13' 24'', \quad B = 79^\circ 46' 38'', \quad AB = 368,$$

to find the rest.

I. To find the side AC.

The angle C is equal to $180^\circ - (A + B) = 55^\circ 59' 58''$, therefore,

As sin. C	$55^\circ 59' 58''$	arith. comp.	0.0814286
: AB	368		2.5658478
:: sin. B	$79^\circ 46' 38''$		9.9930503
: AC	436.844		<u>2.6403267.</u>

II. To find the side CB.

As sin. C	$55^\circ 59' 58''$	arith. comp.	0.0814286
: AB	368		2.5658478
:: sin. A	$44^\circ 13' 24''$		9.8435174
: CB	309.595		<u>2.4907938.</u>

4. In the plane triangle ABC are given

$$AB = 408 \text{ yards, } A = 74^\circ 14', \quad B = 49^\circ 23';$$

to find the other two sides,

$$AC = 371.9 \text{ yards.}$$

$$BC = 416.02 \text{ yards.}$$

5. In the plane triangle ABC are given

$$AB = 408 \text{ yards, } A = 58^\circ 7', B = 22^\circ 37';$$

to find the other two sides,

$$AC = 158.98 \text{ yards}$$

$$BC = 351.03 \text{ yards.}$$

6. In the plane triangle ABC are given

$$AB = 318, BC = 195, A = 32^\circ 40';$$

to find the angle C.

$$C = 61^\circ 40' 3'', \text{ or } 118^\circ 19' 57''.$$

CASE II.

(24.) When two sides and the included angle are given.

RULE, (ART. 19.)

As the sum of the two given sides,

: their difference,

:: tangent of half the sum of the opposite angles

: tangent of half their difference.

Having thus found the half difference of the unknown angles, we obtain the angles themselves, by first adding and then subtracting this half difference from the half sum. The angles being thus known, as well as two sides, the third side is found by the first case.

The student will find a more compendious method of solution for this case in Prob. I., Part IV.; but the rule here given will be more easily remembered.

EXAMPLES.

1. In the triangle ABC are given

$$AB = 137, AC = 153, A = 40^\circ 33' 12'';$$

to find the other parts.

I. To find the other two Angles.

The sum of the other two angles is $(B + C) = 180^\circ - A = 139^\circ 26' 48''$, therefore

As $AB + AC$	290	.	arith. comp.	7.5376020
: $AC \sim AB$	16	.	.	1.2041200
:: $\tan. \frac{1}{2} (B + C)$	$69^\circ 43' 24''$	.	.	10.4324460
: $\tan. \frac{1}{2} (B \sim C)$	$8^\circ 29' 37''$	.	.	9.1741680
By adding	$78^\circ 13' 1''$	=	greater angle B.	
By subtracting	$61^\circ 13' 47''$	=	less angle C.	

II. To find the third side BC.

As $\sin. B$	$78^\circ 13' 1''$	.	arith. comp.	0.0092493
: AC	153	.	.	2.1846914
:: $\sin. A$	$40^\circ 33' 12''$	.	.	9.8130173
: BC	101.616	.	.	2.0069580.

2. In the triangle ABC are given

$$AC = 378, BC = 526, C = 32^\circ 18' 26'';$$

to find the other parts.

I. To find the Angles.

The sum of the angles A, B is $(A + B) = 180^\circ - 32^\circ 18' 26'' = 147^\circ 41' 34''$.

As $AC + BC$	904	.	arith. comp.	7.0438316
: $AC \sim BC$	148	.	.	2.1702617
:: $\tan. \frac{1}{2} (A + B)$	$73^\circ 50' 47''$	.	.	10.5381278
: $\tan. \frac{1}{2} (A \sim B)$	$29^\circ 29' 34''$	.	.	9.7522211.
By adding,	$103^\circ 10' 21''$	=	greater angle B.	
By subtracting,	$44^\circ 31' 13''$	=	less angle A.	

II. To find the side AB.

As sin. A	44° 31' 13"	arith. comp.	0.1541818
: BC	526		2.7209857
:: sin. C	32° 18' 26"		9.7279143
: AB	400.942		<u>2.6030818.</u>

If we wish to obtain the third side of the triangle immediately, without first finding the angle, we may do so by means of the formula at (17), adverted to in the scholium to last case; but as the computation will not be adapted to logarithms, it will in general be the shortest method to proceed as above, by two proportions.

3. In the triangle ABC are given

$$AB = 1637, AC = 2065, A = 132^\circ 7' 12'';$$

to determine the remaining parts.

$$B = 26^\circ 52' 42\frac{1}{4}'', C = 21^\circ 0' 56'', BC = 3387.974.$$

4. In the triangle ABC are given

$$AB = 1686, BC = 960, B = 128^\circ 4';$$

to find the rest.

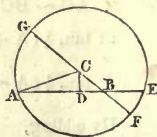
$$A = 18^\circ 21' 20'', C = 33^\circ 34' 40'', AC = 2400.364.$$

CASE III.

(25.) When the three sides are given.

A rule for this case, easy to be remembered, may be deduced from the following simple geometrical investigation.

Take the longest side AB of the triangle for base, and demit upon it the perpendicular CD from the vertex, which will necessarily fall within the base. With centre C and radius CA, equal to the longer of the two sides AC,



CB, describe a circle, and produce the sides AB, BC, to meet the circumference; then it is plain that

$$GB = AC + CB, BF = AC - CB, BE = AD - DB.$$

Now (Geom. Prop. 24, book 6.)

$$GB \cdot BF = AB \cdot BE \therefore AB \cdot (AD - DB) = (AC + CB)(AC - CB)$$

$$\therefore AB : AC + CB :: AC - CB : AD - DB;$$

hence the following rule.

RULE I.

Consider the longest side of the triangle as the base, and demit upon it a perpendicular from the opposite vertex, dividing the base into two segments; then say

As the base,

: the sum of the other two sides,

: : the difference of those sides

: the difference of the segments of the base.

Having thus the sum and difference of the segments, each segment becomes known, and, therefore, in each of the two right-angled triangles into which the proposed is divided, there will be known the base and hypotenuse, and this is enough to determine all the other parts.

RULE II. (ART. 20.)

$$\cos. \frac{1}{2} A = \sqrt{\frac{\frac{1}{2} S (\frac{1}{2} S - a)}{bc}}, \quad \sin. \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2} S - c)(\frac{1}{2} S - b)}{bc}}$$

$$\tan. \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2} S - b)(\frac{1}{2} S - c)}{\frac{1}{2} S (\frac{1}{2} S - a)}}.$$

Both these rules are adapted to logarithmic computation, and this last is much the shortest; when, however, the three sides are small numbers, it will be best to operate without logarithms, by means of the

formula (20),

$$\cos. A = \frac{(b + c + a)(b + c - a)}{2bc} - 1.$$

In applying the logarithmic formulas in Rule 2 to the determination of any particular angle, it will generally be best, when this angle is opposite to the longest side of the triangle, to use the first formula, and when it is opposite to the shortest side to use the second; the third may be used when the required angle is opposite to the mean side. If two sides of the triangle are equal, then, of course, neither of these formulas will be used, as the unknown parts will be more readily found as in Example 3, p. 15.

EXAMPLES.

1. The three sides of the triangle ABC are

$$AB = 1637, AC = 2065, BC = 3387.974;$$

required the angle A.

$$a = 3387.974$$

$$b = 2065 \text{ arith. comp. } 6.6850799$$

$$c = 1637 \text{ arith. comp. } 6.7859513$$

$$\hline 2)7089.974$$

$$\frac{1}{2} S = 3544.987 \quad . \quad 3.5496146$$

$$\frac{1}{2} S - a = 157.013 \quad . \quad 2.1959356$$

$$\hline 2)19.2165814$$

$$\cos. \frac{1}{2} A = 66^\circ 3' 36'' \quad . \quad 9.6082907$$

$$\therefore A = 132^\circ 7' 12''.$$

2. The three sides of the triangle ABC are

$$AB = 98, BC = 95.12, AC = 162.34;$$

to determine the angle A.

Using the third formula in the second rule, we have

$$a = 95.12$$

$$b = 162.34$$

$$c = 98$$

$$2) 355.46$$

$$S = 177.73 \quad \text{arith. comp. } 7.7502393$$

$$\frac{1}{2} S - a = 82.61 \quad \text{arith. comp. } 8.0829674$$

$$\frac{1}{2} S - b = 15.39 \quad . \quad 1.1872386$$

$$\frac{1}{2} S - c = 79.73 \quad . \quad 1.9016218$$

$$2) 18.9220671$$

$$\tan. \frac{1}{2} A = 16^\circ 7' 26\frac{2}{3}'' \quad . \quad 9.4610335$$

$$\therefore A = 32^\circ 14' 53\frac{1}{3}''.$$

3. In the triangle ABC are given

$$AC = 6, \quad AB = 5.523, \quad BC = 1.372;$$

required the angle A.

Applying the second formula to this example we have

$$a = 1.372$$

$$b = 6 \quad \text{arith. comp. } 9.2218487$$

$$c = 5.523 \quad \text{arith. comp. } 9.2578250$$

$$2) 12.895$$

$$\frac{1}{2} S = 6.4475$$

$$\frac{1}{2} S - b = .4475 \quad . \quad \bar{1}.6577930$$

$$\frac{1}{2} S - c = .9245 \quad . \quad \bar{1}.9659069$$

$$2) 18.0963736$$

$$\sin. \frac{1}{2} A = 6^\circ 24' 55'' \quad . \quad 9.0481868$$

$$\therefore A = 12^\circ 49' 50''.$$

4. The three sides of a plane triangle are

$$AB = 137, AC = 153, BC = 101.616;$$

required the three angles,

$$A = 40^{\circ} 33' 12'', B = 78^{\circ} 13' 1'', C = 61^{\circ} 13' 47''.$$

5. The three sides of a plane triangle are

$$AB = 1686, BC = 960, AC = 2400.364;$$

required the angle B,

$$B = 128^{\circ} 4'.$$

6. Required the angles when the sides are 4, 5, and 6.

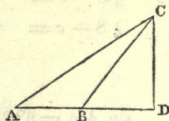
The angles are $41^{\circ} 24' 35''$, $55^{\circ} 46' 16''$, and $82^{\circ} 49' 9''$.

CHAPTER III.

APPLICATION OF PLANE TRIGONOMETRY TO THE MENSURATION OF HEIGHTS AND DISTANCES.

PROBLEM I.

A person on one side of a river observes an obelisk on the opposite side, and, being desirous to ascertain its height, he took with a quadrant the angle $B = 55^{\circ} 54'$, which the obelisk subtended at the place where he stood, then going back the distance $BA = 100$ feet, he again measured the subtended angle, and found it to be $A = 33^{\circ} 20'$; what was the height of the obelisk?



In the triangle of ABC are given the angle $A = 33^\circ 20'$, the angle $ACB = 55^\circ 54' - 33^\circ 20' = 22^\circ 34'$, and the side AB; and, therefore, BC may be found by Case 1. of oblique angled triangles. Again, in the triangle BCD, we shall have given the side BC, and the angle B to find CD, which belongs to Case 1. of right angled triangles.

The actual computation, however, will be shortened by combining these two rules in a single formula, thus from the first

$$BC = \frac{AB \sin. A}{\sin. ACB},$$

and from the second

$$CD = BC \sin. CBD$$

$$\therefore CD = \frac{AB \sin. A \sin. CBD}{\sin. ACB}$$

$$\sin. ACB \quad 22^\circ 34' \text{ arith. comp. } 0.4159424$$

$$\sin. A \quad 33^\circ 20' \quad . \quad 9.7399748$$

$$\sin. CBD \quad 55^\circ 54' \quad . \quad . \quad 9.9186620$$

$$AB \quad 100 \quad . \quad . \quad 2$$

$$\hline CD \quad 118.57^* \quad . \quad . \quad 2.0739792.$$

The problem may be solved still more readily, as follows.

If we take CD for radius, DB will be the tangent of the angle DCB, and DA the tangent of DCA, therefore, AB is the difference of those tangents; but by referring to the table of natural tangents, we find that to radius 1

$$\text{nat. tan. } 56^\circ 40' = 1.5204261$$

$$\text{nat. tan. } 34^\circ 6' = 6770509$$

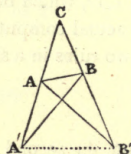
$$\hline \text{difference} = .8433752$$

$$\therefore .8433752 : 1 :: 100 : 118.57, \text{ as before.}$$

* To the height of the object thus determined the height of the observer's eye, or of the instrument, must be added.

PROBLEM II.

A person at A wishes to know his distance from an inaccessible object at C, but he has no instrument for taking angles. He, therefore, sets up a staff at A, from which he measures the distance $AA' = 60$ feet, so that when he stands at A' the staff and the object appear in the same straight line A'C; he, in like manner, measures another distance $BB' = 86$ feet, from a second station B, 38 feet from the former A, and he finds the diagonal distances AB' , BA' , to be respectively 97 feet and 81 feet. From these data it is required to determine the distance of A from the object C



All the three sides of the triangle A'AB are given, therefore to find the angle A'AB we have, by using the first formula at (25),

$$A'B = 81$$

$$A'A = 60 \text{ arith. comp. } 8.2218487$$

$$AB = 38 \text{ arith. comp. } 8.4202164$$

$$\hline 2)179$$

$$\frac{1}{2} S = 89.5 \quad . \quad . \quad 1.9518230$$

$$\frac{1}{2} S - A'B = 8.5 \quad . \quad . \quad 0.9294189$$

$$\hline 2)19.5233070$$

$$\cos. \frac{1}{2} A AB = 54^\circ 41' 56'' \quad . \quad . \quad 9.7616535$$

$$\therefore A'AB = 109^\circ 23' 52'' \therefore CAB = 70^\circ 36' 8''.$$

Again, by applying the same formula to the triangle B'BA, we have

$$B'A = 97$$

$$B'B = 86 \text{ arith. comp. } 8.0655015$$

$$AB = 38 \text{ arith. comp. } 8.4202164$$

$$\begin{array}{r} 2)221 \\ \hline \end{array}$$

$$110.5 \quad . \quad 2.0433623$$

$$13.5 \quad . \quad 1.1303338$$

$$\begin{array}{r} 2)19.6594140 \\ \hline \end{array}$$

$$\cos. \frac{1}{2} B'BA = 47^{\circ} 29' 50'' \quad . \quad 9.8297070$$

$$\therefore B'BA = 94^{\circ} 59' 40'' \quad \therefore CBA = 85^{\circ} 0' 20''$$

$$\therefore C = 180 - (CAB + CBA) = 24^{\circ} 23' 40''.$$

Consequently, in the triangle ABC, we have all the angles and one side AB given; hence, by Case I.

$$\sin. C \quad 24^{\circ} 23' 40'' \text{ arith. comp. } 0.3840330$$

$$: AB \quad 38 \quad . \quad 1.5797836$$

$$:: \sin. B \quad 85^{\circ} 0' 20'' \quad . \quad 9.9983479$$

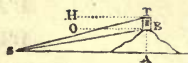
$$\therefore AC \quad 91.657 \quad . \quad 1.9621645.$$

PROBLEM III.

At the top of a castle, which stood on a hill near the sea-shore, the angle of depression HTS, of a ship at anchor, was observed to be $4^{\circ} 52'$; at the bottom of the castle the angle of depression OBS was $4^{\circ} 2'$. Required the horizontal distance AS of the vessel, and the height of the hill above the level of the sea, the height of the castle being 60 feet.

As TH, BO, are parallel to AS, we have $TSA = 4^{\circ} 52'$, and $BSA = 4^{\circ} 2'$. Bearing this in mind we have

$$\text{In } \triangle \text{ TSB, } \frac{\text{SB}}{\text{BT}} = \frac{\sin. \text{ ATS}}{\sin. \text{ TSB}}$$



In $\triangle BSA$, $AS = SB \cos. BSA$, $AB = SB \sin. BSA$

$$\therefore AS \cdot R = \frac{BT \sin. ATS \cos. BSA}{\sin. TSB}$$

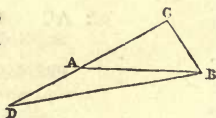
$$AB \cdot R = \frac{BT \sin. ATS \sin. BSA}{\sin. TSB};$$

hence the logarithmic operation will be

sin. TSB	0° 50' arith. comp.	1·8373192	.	.	1·8373192
sin. ATS	85° 8'	9·9984315	.	.	9·9984315
cos. BSA	4° 2'	9·9989230	sin. BSA	.	8·8471827
BT	60	1·7781513	.	.	1·7781513
AS	4100·4	3·6128250,	AB	289·12	2·4610847

PROBLEM IV.

The distances of three objects A, B, C, from each other, are as follow, viz. $AB = 462$ yards, $AC = 328$ yards, and $BC = 297$ yards; a person at D, wishing to know his distance from each object, takes the angle ADB , and finds it to be $34^{\circ} 16' 21''$; it is required to determine DA , DC , and DB .



As the three sides of the triangle ABC are given we may find the angle CAB, and, consequently, the supplemental angle DAB, so that we shall have in the triangle DAB the two angles D, A and the side AB to find the rest. The computation will, therefore, be as follows.

I. To find the angle CAB.

$$BC = 297$$

$$AC = 328 \text{ arith. comp. } 7.4841262$$

$$AB = 462 \text{ arith. comp. } 7.3353580$$

$$\begin{array}{r} 2)1087 \\ \hline \end{array}$$

$$543.5$$

$$215.5 \quad . \quad 2.3334473$$

$$81.5 \quad . \quad 2.9111576$$

$$\begin{array}{r} 2)19.0640891 \\ \hline \end{array}$$

$$\sin. \frac{1}{2} CAB = 19^{\circ} 54' 14'' \quad . \quad 9.5320445$$

$$\therefore DAB = 180^{\circ} - 39^{\circ} 48' 28'' = 140^{\circ} 11' 32'' \therefore DBA = 15^{\circ} 32' 7''.$$

II. To find AD.

$$\text{As sin. D } 24^{\circ} 16' 21'' \text{ arith. comp. } 0.3860770$$

$$: AB \quad 462 \quad . \quad . \quad 2.6646420$$

$$: : \sin. B \quad 15^{\circ} 32' 7'' \quad . \quad . \quad 9.4278619$$

$$: AD \quad 301.01 \quad . \quad . \quad 2.4785809$$

$$\therefore DC = DA + AC = 629.101 \text{ yards.}$$

III. To find BD.

$$\text{As sin. D } 24^{\circ} 16' 21'' \text{ arith. comp. } 0.3860770$$

$$: AB \quad 462 \quad . \quad . \quad 2.6646420$$

$$: : \sin. A \quad 140^{\circ} 11' 32'' \quad . \quad . \quad 9.8063252$$

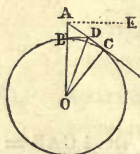
$$: BD \quad 719.522 \quad . \quad . \quad 2.8570442.$$

Hence the three distances are $DA = 301.01$, $DC = 629.101$, $DB = 719.522$.

PROBLEM V.

Suppose that from the top of a mountain, three miles high, the angle of depression of the remotest visible point of the earth's surface is taken and found to be $2^\circ 13' 27''$; it is required thence to determine the diameter of the earth, supposing it to be a perfect sphere.

Let O be the centre of the earth, BA the mountain, AC the visual ray or line touching the earth's surface in C . Draw the tangent BD , and join OD , OC ; then the angle of depression EAC being given, we have also the angle BAD , the complement of it, equal to $87^\circ 46' 33''$. Also since the tangents BD , CD , are equal, (Geom. p. 105,) we have the angle $BOD = DOC = \frac{1}{2}$ comp. $A = 1^\circ 6' 49\frac{1}{2}''$, and, therefore, $BDO = 88^\circ 53' 16\frac{1}{2}''$.



Now in the right-angled triangle ABD we have

$$BD = AB \tan. A;$$

and in the right-angled triangle OBD ,

$$OB = BD \tan. BDO;$$

hence, by substitution,

$$OB = AB \tan. A \tan. BDO;$$

the computation is, therefore, as follows:

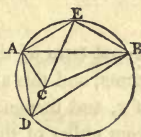
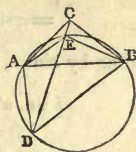
$AB =$	3	.	0.4771213
$\tan. A$	$87^\circ 46' 33''$	.	11.4107381
$\tan. BDO$	$88^\circ 53' 16\frac{1}{2}''$	.	11.7119309
OB	3979.15	.	<u>3.5997903;</u>

hence the diameter is 7958.3 miles.

PROBLEM VI.

Given the distances between three objects A, B, C, and the angles subtended by these distances at a point D in the same plane with them ; to determine the distance of D from each object.

Let a circle be described about the triangle ADB, and join AE, EB, then will the angles ABE, BAE, be respectively equal to the given angles ADE, BDE, (Geom. p. 50;) thus all the angles of the triangle AEB are known, as also the side AB; we may find, therefore, the remaining sides AE, EB. Again, the sides of the triangle ABC being known, we may find the angle BAC; hence the angle CAE becomes known, so that in the triangle CAE we shall have the two sides AE, AC, and the included angle given, from which we may find the angle AEC in fig. 1, or the angle ACE in fig. 2, and thence its supplement AED or ACD; this with the given side AE and angle ADE, in the first figure, or with the given side AC and angle ADC in the second, will enable us to find AD, one of the required lines, and thence DC and DB the other two.



Or the solution may be conducted more analytically as follows.

Put x for the angle DAC, and x' for the angle DBC; also call the given angles ADC, BDC, α and α' , then a, b, c , representing as usual the sides opposite to A, B, C, we have

$$\frac{\sin. \alpha}{\sin. x} = \frac{b}{DC}, \quad \frac{\sin. \alpha'}{\sin. x'} = \frac{a}{DC} \dots (1)$$

$$\therefore \frac{\sin. \alpha \sin. x'}{\sin. \alpha' \sin. x} = \frac{b}{a} \therefore a \sin. \alpha \sin. x' = b \sin. \alpha' \sin. x \dots (2).$$

This is one equation between the unknown quantities x, x' . Another is easily obtained; for since the four angles of the quadrilateral ADCB make up four right angles or 360° , we have $x + x' + \alpha + \alpha' + ACD$

$+BCD = 360^\circ$; the two latter angles may be considered as known, since in the triangle ABC the angle C is determinable from the three given sides; therefore all the terms in the first member of this equation are known except x and x' . Call the sum of these known quantities β , and we shall thus have $x' = \beta - x$, and, consequently, by substitution, equation (2) becomes

$$\begin{aligned} a \sin. a \sin. (\beta - x) &= b \sin. a' \sin. x \\ &= a \sin. a (\sin. \beta \cos. x - \cos. \beta \sin. x); \end{aligned}$$

or dividing by $\sin. x$,

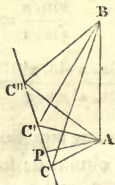
$$\begin{aligned} b \sin. a' &= a \sin. a (\sin. \beta \cot. x - \cos. \beta) \\ \therefore \cot. x &= \frac{b \sin. a'}{a \sin. a \sin. \beta} + \frac{\cos. \beta}{\sin. \beta} \\ &= \frac{b \sin. a'}{a \sin. a \sin. \beta} + \cot. \beta. \end{aligned}$$

The first term of this second member may be easily calculated by logarithms, and this added to the natural cotangent of β gives the nat. cot. of x , and thence x' is known from the equation $x' = \beta - x$, and CD from either of the equations (1).

PROBLEM VII.

Given the angles of elevation of an object taken at three places on the same horizontal straight line, together with the distances between the stations; to find the height of the object and its distance from either station.

Let AB be the object, and C, C', C'', the three stations, then the triangles BCA, BC'A, BC''A, will all be right angled at A; and, therefore, to radius BA, AC, AC', AC'', will be the tangents of the angles at B, or the cotangents of the angles of elevation; hence putting a, a', a'' , for the angles of elevation, x for the height of the object, and a, b , for the distances C C',



$C'' C'''$, we shall have

$$AC = x \cot. \alpha, AC' = x \cot. \alpha', AC'' = x \cot. \alpha''.$$

Now if a perpendicular AP be drawn from A to $C C''$, we shall have (Geom. p. 35,) from the triangle ACC

$$AC^2 = AC'^2 + C' C^2 - 2 C' C \cdot C' P;$$

and from the triangle $AC' C''$

$$AC''^2 = AC'^2 + C'' C'^2 + 2 C'' C' \cdot C' P;$$

that is, we shall have the two equations

$$x^2 \cot.^2 \alpha = x^2 \cot.^2 \alpha' + a^2 - 2 a \cdot C' P$$

$$x^2 \cot.^2 \alpha'' = x^2 \cot.^2 \alpha' + b^2 + 2 b \cdot C' P.$$

In order to eliminate $C' P$, multiply the first by b , the second by a , and add, and we shall have

$$x^2 (b \cot.^2 \alpha + a \cot.^2 \alpha'') = (a + b) x^2 \cot.^2 \alpha' + ab (a + b)$$

$$\therefore x = \sqrt{\frac{ab (a + b)}{b \cot.^2 \alpha + a \cot.^2 \alpha'' - (a + b) \cot.^2 \alpha'}}.$$

If the three stations are equidistant, then $a = b$, and the expression becomes

$$x = \frac{a}{\sqrt{\frac{1}{2} \cot.^2 \alpha + \frac{1}{2} \cot.^2 \alpha'' - \cot.^2 \alpha'}}.$$

The height AB being thus determined, the distances of the stations from the object are found by multiplying this height by the cotangents of the angles of elevation.

PROBLEM VIII.

Three objects A, B, and C, whose distances are $AC = 8$ miles, $BC = 7\frac{1}{2}$ miles, and $AB = 12$ miles, are visible from one station D, in the line joining A and B, at which point the line joining A and C subtends

an angle of $107^{\circ} 56' 13''$. Required the distances of the objects from the station,

$$AD = 5 \text{ miles, } DC = 4.892 \text{ miles, } DB = 7 \text{ miles.}$$

PROBLEM IX.

Suppose the angle of elevation of the top of a steeple to be 40° when the observer's eye is level with the bottom, and that from a window 18 feet, directly above the first station, the angle of elevation is found to be $37^{\circ} 30'$. Required the height and distance of the steeple.

$$\text{Height} = 210.44 \text{ feet. Distance } 250.79 \text{ feet.}$$

PROBLEM X.

In order to determine the horizontal distance between two remote objects A, B, a base line A'B' of 536 yards was measured, and then a flagstaff being set up at each extremity, these four angles were taken from them, viz. at A' the angular distance between A and B, $57^{\circ} 40'$, and the angular distance between B and B', $40^{\circ} 16'$, also at B' the angular distance between A and B, $71^{\circ} 7'$, and the angular distance between A' and A, $42^{\circ} 22'$. Required the distance between the objects.

$$939.52 \text{ yards.}$$

PROBLEM XI.

Three objects A, B, C, are in the same straight line, and of known distances from each other, viz. $AB = 3.626$ yards, and $BC = 8.374$ yards, the angular distance of A, B, from a station D, where all the objects are visible, is 19° , and the angular distance of B, C, is 25° . Required the distance of each object from the place of observation.

$$DA = 9.471 \text{ yards, } DB = 10.861, DC = 16.848.$$

PROBLEM XII.

At three points in the same horizontal straight line the angles of elevation of an object ^{were} found to be $36^\circ 50'$, $21^\circ 24'$ and 14° , the middle station being 84 feet from each of the others. Required the height of the object.

53.964 feet. *ans.* 27.23 ft.
~~53.96084~~

PROBLEM XIII.

There are three towns A, B, and C, whose distances apart are as follow: from A to B, 6 miles; from A to C, 22 miles; and from B to C, 20 miles. A messenger is despatched from B to A, and has to call at a town D in a direct line between A and C. Now in travelling from B to D, he walks uniformly at the rate of 4 miles an hour, and from D to A at the rate of 3 miles an hour. Supposing him to perform his journey in 3 hours, it is required to determine the position of the town D.

The distance of D from A is 4.72 miles.

The student who has the practical applications of Plane Trigonometry more immediately in view may pass over the following chapter, on the theory of the trigonometrical lines, and proceed to the first chapter of part III., which contains the application of Trigonometry to Navigation.

CHAPTER IV.

INVESTIGATION OF TRIGONOMETRICAL FORMULAS.

(26.) The formulas hitherto investigated are those only which are immediately connected with the business of plane trigonometry, properly so called, that is, with the solutions of the several cases of plane triangles. Having disposed of all these cases, we shall now proceed to develop the theory of the trigonometrical lines more at large, dismissing all considerations of the sides of triangles.

The following general expressions have already been established, viz.

$$\left. \begin{aligned} \sin. (A + B) &= \sin. A \cos. B + \sin. B \cos. A \\ \sin. (A - B) &= \sin. A \cos. B - \sin. B \cos. A \end{aligned} \right\} \dots \dots (1)$$

$$\left. \begin{aligned} \cos. (A + B) &= \cos. A \cos. B - \sin. A \sin. B \\ \cos. (A - B) &= \cos. A \cos. B + \sin. A \sin. B \end{aligned} \right\} \dots \dots (2).$$

From these equations we get

1. By addition,

$$\left. \begin{aligned} \sin. (A + B) + \sin. (A - B) &= 2 \sin. A \cos. B \\ \cos. (A + B) + \cos. (A - B) &= 2 \cos. A \cos. B \end{aligned} \right\} \dots \dots (3).$$

2. By subtraction,

$$\left. \begin{aligned} \sin. (A + B) - \sin. (A - B) &= 2 \cos. A \sin. B \\ \cos. (A - B) - \cos. (A + B) &= 2 \sin. A \sin. B \end{aligned} \right\} \dots \dots (4).$$

It is worth while to remark here that if we make $A = 60^\circ$, then since $\cos. 60^\circ = \frac{1}{2}$, (p. 9,) the first of these formulas furnish the equation

$$\sin. B = \sin. (60^\circ + B) - \sin. (60^\circ - B) \dots \dots (V);$$

which is a useful expression in the work of computing tables.

3. By multiplication,

$$\sin. (A + B) \sin. (A - B) = \sin.^2 A \cos.^2 B - \sin.^2 B \cos.^2 A$$

$$\cos. (A + B) \cos. (A - B) = \cos.^2 A \cos.^2 B - \sin.^2 A \sin.^2 B.$$

Or eliminating $\cos.^2 A$, $\cos.^2 B$, from each of these equations by means of the conditions

$$\sin.^2 A + \cos.^2 A = 1$$

$$\sin.^2 B + \cos.^2 B = 1;$$

the second members of them become, respectively,

$$\sin.^2 A - \sin.^2 A \sin.^2 B - \sin.^2 B + \sin.^2 B \sin.^2 A, \text{ or } \sin.^2 A - \sin.^2 B;$$

and,

$$1 - \sin.^2 B - \sin.^2 A + \sin.^2 A \sin.^2 B - \sin.^2 A \sin.^2 B, \text{ or } \cos.^2 B - \sin.^2 A;$$

so that

$$\left. \begin{aligned} \sin. (A + B) \sin. (A - B) &= \sin.^2 A - \sin.^2 B = (\sin. A + \sin. B)(\sin. A - \sin. B) \\ \cos. (A + B) \cos. (A - B) &= \cos.^2 B - \sin.^2 A = (\cos. B + \sin. A)(\cos. B - \sin. A) \end{aligned} \right\} (5)$$

4. By division,

$$\frac{\sin. (A + B)}{\sin. (A - B)} = \frac{\sin. A \cos. B + \sin. B \cos. A}{\sin. A \cos. B - \sin. B \cos. A}$$

$$\frac{\cos. (A + B)}{\cos. (A - B)} = \frac{\cos. A \cos. B - \sin. A \sin. B}{\cos. A \cos. B + \sin. A \sin. B}$$

$$\frac{\sin. (A \pm B)}{\cos. (A \pm B)} = \frac{\sin. A \cos. B \pm \sin. B \cos. A}{\cos. A \cos. B \mp \sin. A \sin. B}$$

The right-hand members of these equations will assume other useful forms by dividing both numerator and denominator of each by certain expressions: thus, let the divisors for the first equation be

$$\cos. A \cos. B, \sin. A \sin. B, \sin. A \cos. B;$$

those for the second,

$$\cos. A \sin. B, \sin. A \cos. B, \cos. A \cos. B;$$

and those for the third the same as those for the first; we shall then have

$$\left. \begin{aligned} \frac{\sin.(A+B)}{\sin.(A-B)} &= \frac{\tan.A + \tan.B}{\tan.A - \tan.B} = \frac{\cot.B + \cot.A}{\cot.B - \cot.A} = \frac{1 + \cot.A \tan.B}{1 - \cot.A \tan.B} \\ \frac{\cos.(A+B)}{\cos.(A-B)} &= \frac{\cot.B - \tan.A}{\cot.B + \tan.A} = \frac{\cot.A - \tan.B}{\cot.A + \tan.B} = \frac{1 - \tan.A \tan.B}{1 + \tan.A \tan.B} \\ \frac{\sin.(A \pm B)}{\cos.(A \pm B)} &= \frac{\tan.A \pm \tan.B}{1 \mp \tan.A \tan.B} = \frac{\cot.B \pm \cot.A}{\cot.A \cot.B \mp 1} = \frac{1 \pm \cot.A \tan.B}{\cot.A \mp \tan.B} \end{aligned} \right\} (6).$$

The last of these immediately gives

$$\left. \begin{aligned} \tan.(A+B) &= \frac{\tan.A + \tan.B}{1 - \tan.A \tan.B}, \quad \tan.(A-B) = \frac{\tan.A - \tan.B}{1 + \tan.A \tan.B} \\ \cot.(A+B) &= \frac{\cot.A \cot.B - 1}{\cot.B + \cot.A}, \quad \cot.(A-B) = \frac{\cot.A \cot.B + 1}{\cot.B - \cot.A} \end{aligned} \right\} (7).$$

If $A = 45^\circ$, then $\tan.A = \cot.A = 1$, therefore,

$$\tan.(45^\circ + B) = \frac{1 + \tan.B}{1 - \tan.B}, \quad \tan.(45^\circ - B) = \frac{1 - \tan.B}{1 + \tan.B}$$

$$\cot.(45^\circ + B) = \frac{\cot.B - 1}{\cot.B + 1}, \quad \cot.(45^\circ - B) = \frac{\cot.B + 1}{\cot.B - 1}$$

$$\therefore \tan.(45^\circ + B) - \tan.(45^\circ - B) = \frac{4 \tan.B}{1 - \tan.^2 B} \dots (8)$$

$$\cot.(45^\circ - B) - \cot.(45^\circ + B) = \frac{4 \cot.B}{\cot.^2 B - 1} \dots (9);$$

Such are the most useful theorems respecting the sums and differences of two unequal arcs, and they may be converted into other expressions, involving three or more arcs by simply substituting $B + C + D + \&c.$ for B . We shall briefly consider the case of three arcs, or angles, because of a curious property belonging to them whenever they make up either 180° or 90° .

Let A, B, C , be any three arcs, and consider $A + B$ as one, then by equa. (1)

$$\begin{aligned} \sin.(A+B+C) &= \sin.(A+B) \cos.C + \cos.(A+B) \sin.C \\ &= (\sin.A \cos.B + \cos.A \sin.B) \cos.C + \\ &\quad (\cos.A \cos.B - \sin.A \sin.B) \sin.C \\ \cos.(A+B+C) &= \cos.(A+B) \cos.C - \sin.(A+B) \sin.C \\ &= (\cos.A \cos.B - \sin.A \sin.B) \cos.C - \\ &\quad (\sin.A \cos.B + \cos.A \sin.B) \sin.C. \end{aligned}$$

Let now the sum of the three arcs be 180° , or, indeed, any multiple of 180° , then the sine of this sum will be 0, so that the first of these equations gives

$$\begin{aligned} \sin. A \cos. B \cos. C + \cos. A \sin. B \cos. C + \\ \cos. A \cos. B \sin. C = \sin. A \sin. B \sin. C; \end{aligned}$$

dividing both sides of this equation by $\cos. A \cos. B \cos. C$, we have

$$\frac{\sin. A}{\cos. A} + \frac{\sin. B}{\cos. B} + \frac{\sin. C}{\cos. C} = \frac{\sin. A}{\cos. A} \cdot \frac{\sin. B}{\cos. B} \cdot \frac{\sin. C}{\cos. C};$$

that is,

$$\tan. A + \tan. B + \tan. C = \tan. A \tan. B \tan. C;$$

a remarkable property of the angles of a plane triangle.

Again, let the sum of the three arcs be 90° , or any multiple thereof, then the cosine of this sum will be 0, so that the second general equation above becomes

$$\begin{aligned} \cos. A \cos. B \cos. C = \cos. A \cos. B \cos. C + \\ \sin. A \cos. B \sin. C + \cos. A \sin. B \sin. C; \end{aligned}$$

dividing both sides by $\sin. A \sin. B \sin. C$, we have

$$\cot. A \cot. B \cot. C = \cot. A + \cot. B + \cot. C.$$

(28.) To deduce formulas for multiple arcs we have only to put nA for $A + B$ in the preceding expressions. We thus get from (1)

$$\begin{aligned} \sin. nA &= \sin. A \cos. (n-1)A + \sin. (n-1)A \cos. A \\ \cos. nA &= \cos. A \cos. (n-1)A - \sin. A \sin. (n-1)A; \end{aligned}$$

so that putting for n , 1, 2, 3, &c. successively, we have

$$\left. \begin{aligned} \sin. A &= \sin. A \\ \sin. 2A &= 2 \sin. A \cos. A \\ \sin. 3A &= \sin. A \cos. 2A + \sin. 2A \cos. A \\ \sin. 4A &= \sin. A \cos. 3A + \sin. 3A \cos. A \\ &\quad \&c. \qquad \qquad \&c. \end{aligned} \right\} \dots (10)$$

$$\left. \begin{aligned}
 \cos. A &= \cos. A \\
 \cos. 2 A &= \cos.^2 A - \sin.^2 A \\
 \cos. 3 A &= \cos. A \cos. 2 A - \sin. A \sin. 2 A \\
 \cos. 4 A &= \cos. A \cos. 3 A - \sin. A \sin. 3 A \\
 &\quad \&c. \qquad \qquad \&c.
 \end{aligned} \right\} \dots (11).$$

We may put the general expressions for $\sin. nA$, and $\cos. nA$, under a different form, by making use of the second equation in (1) and (2), thus putting $(n-1)A$ for A , and A for B , these become

$$\begin{aligned}
 \sin. (n-2) A &= \sin. (n-1) A \cos. A - \sin. A \cos. (n-1) A \\
 \cos. (n-2) A &= \cos. (n-1) A \cos. A + \sin. (n-1) A \sin. A;
 \end{aligned}$$

or, by transposing,

$$\begin{aligned}
 0 &= -\sin. A \cos. (n-1) A + \sin. (n-1) A \cos. A - \sin. (n-2) A \\
 0 &= \cos. A \cos. (n-1) A + \sin. A \sin. (n-1) A - \cos. (n-2) A;
 \end{aligned}$$

adding these two equations to those above, there results

$$\left. \begin{aligned}
 \sin. n A &= 2 \sin. (n-1) A \cos. A - \sin. (n-2) A \\
 \cos. n A &= 2 \cos. (n-1) A \cos. A - \cos. (n-2) A
 \end{aligned} \right\} \dots (12);$$

hence,

$$\left. \begin{aligned}
 \sin. A &= \sin. A \\
 \sin. 2 A &= 2 \sin. A \cos. A \\
 \sin. 3 A &= 2 \sin. 2 A \cos. A - \sin. A \\
 \sin. 4 A &= 2 \sin. 3 A \cos. A - \sin. 2 A \\
 &\quad \&c. \qquad \qquad \&c.
 \end{aligned} \right\} \dots (13)$$

$$\left. \begin{aligned}
 \cos. A &= \cos. A \\
 \cos. 2 A &= 2 \cos. A \cos. A - 1 \\
 \cos. 3 A &= 2 \cos. 2 A \cos. A - \cos. A \\
 \cos. 4 A &= 2 \cos. 3 A \cos. 2 A - \cos. 2 A \\
 &\quad \&c. \qquad \qquad \&c.
 \end{aligned} \right\} \dots (14).$$

(29.) The sines and cosines of multiple arcs may also be developed in terms of the powers of the sine and cosine of the simple arc, by help of a remarkable formula, known by the name of De Moivre's formula, which may be easily established, as follows.

Multiply together the two expressions,

$$\cos. A + \sin. A \cdot \sqrt{-1} \text{ and } \cos. A_1 + \sin. A_1 \cdot \sqrt{-1};$$

and we shall have the product

$$\begin{aligned} &\cos. A \cos. A_1 - \sin. A \sin. A_1 + \\ &(\cos. A \sin. A_1 + \sin. A \cos. A_1) \sqrt{-1}; \end{aligned}$$

which, by the equations (1), (2), is the same as

$$\cos. (A + A_1) + \sin. (A + A_1) \sqrt{-1};$$

which is of the same form as the original factors, consequently, multiplying this by the new factor

$$\cos. A_2 + \sin. A_2 \cdot \sqrt{-1},$$

we must have for the product

$$\cos. (A + A_1 + A_2) + \sin. (A + A_1 + A_2) \sqrt{-1},$$

and thus by continually introducing a new factor, we must have generally

$$\begin{aligned} &(\cos. A + \sin. A \cdot \sqrt{-1}) (\cos. A_1 + \sin. A_1 \cdot \sqrt{-1}) (\cos. A_2 + \\ &\sin. A_2 \cdot \sqrt{-1}) \&c. = \end{aligned}$$

$$\cos. (A + A_1 + A_2 + \&c.) + \sin. (A + A_1 + A_2 + \&c.) \sqrt{-1}.$$

Suppose now that

$$A = A_1 = A_2 = \&c.$$

then this equation will become

$$(\cos. A + \sin. A \cdot \sqrt{-1})^n = \cos. n A + \sin. n A \cdot \sqrt{-1}$$

or, writing the radical with the double sign,

$$(\cos. A \pm \sin. A \cdot \sqrt{-1})^n = \cos. n A \pm \sin. n A \cdot \sqrt{-1} \dots (15);$$

n is here a whole number, but, in order to show that the formula holds when the exponent is a fraction, put $a = \frac{n}{m} A$; then, by this formula,

$$\begin{aligned} (\cos. a \pm \sin. a \cdot \sqrt{-1})^m &= \cos. ma \pm \sin. m A \cdot \sqrt{-1} = \\ \cos. n A \pm \sin. n A \cdot \sqrt{-1} &= (\cos. A \pm \sin. A \cdot \sqrt{-1})^n; \end{aligned}$$

therefore, extracting the m th root of the first and last members, and restoring the value of a , we have

$$\cos. \frac{n}{m} A \pm \sin. \frac{n}{m} A \cdot \sqrt{-1} = (\cos. A \pm \sin. A \cdot \sqrt{-1})^{\frac{n}{m}}. \quad (16);$$

which is the formula of *De Moivre*.

If we take the reciprocal of each side of this equation we shall have

$$\frac{1}{\cos. \frac{n}{m} A \pm \sin. \frac{n}{m} A \cdot \sqrt{-1}} = (\cos. A \pm \sin. A \cdot \sqrt{-1})^{-\frac{n}{m}};$$

and if we multiply both numerator and denominator of the first member of this by $\cos. \frac{n}{m} A \mp \sin. \frac{n}{m} A \cdot \sqrt{-1}$, the denominator will then become $\cos.^2 \frac{n}{m} A + \sin.^2 \frac{n}{m} A = 1$; hence

$$\cos. \frac{n}{m} A \mp \sin. \frac{n}{m} A \cdot \sqrt{-1} = (\cos. A \pm \sin. A \cdot \sqrt{-1})^{-\frac{n}{m}} \quad (17);$$

so that the formula (16) remains true, whether $\frac{n}{m}$ be positive or negative.

If in (16) we make $\frac{n}{m}$ negative, the signs $\pm$, in the first member, will be inverted as here, because the sign of the sine is the same as that of the arc.

It may seem to the student that there is a want of generality in the first members of (16) and (17), which ought to contain m values, seeing that the m th root appears in the second members. But this defect is only apparent; for it must be remembered that while the lines $\sin. A$, $\cos. A$, in the second member have each a certain fixed value, the arcs

A, to which these lines indifferently belong are innumerable. The first member involves a proposed fractional part, not of any particular one of these arcs, but of any one of them indifferently; it is easy to see, therefore, that the first member involves a variety of values, and they may be shown to be in number m . (See note A at the end.)

(30.) Let the first side of (15) be developed by the binomial theorem and the equation will become

$$\cos.^n A \pm n \cos.^{n-1} A p + \frac{n(n-1)}{2} \cos.^{n-2} A p^2 \pm \&c.$$

$$= \cos. n A \pm \sin. n A \cdot \sqrt{-1};$$

p being put for the imaginary $\sin. A \cdot \sqrt{-1}$.

Now as in any equation the imaginaries on one side are together equal to those on the other, (Alg. p. 126,) we have by expunging all the imaginaries on both sides, the following expression for $\cos. nA$, viz.

$$\begin{aligned} \cos. n A = \cos.^n A - \frac{n(n-1)}{2} \cos.^{n-2} A \sin.^2 A + \\ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} \cos.^{n-4} A \sin.^4 A - \&c. \end{aligned}$$

In like manner, by expunging all the rational terms on each side of the same equation, and then dividing by $\sqrt{-1}$, there results for $\sin. nA$,

$$\begin{aligned} \sin. n A = n \cos.^{n-1} A \sin. A - \frac{n(n-1)(n-2)}{2 \cdot 3} \cos.^{n-3} A \sin.^3 A \\ + \&c. \end{aligned}$$

From these two expressions may be obtained series, for the value of the sine and cosine of an arc in terms of the arc itself. For let $n = \frac{1}{0}$,

and $\sin. A = 0 = A$, then $nA = \frac{0}{0} =$ any finite quantity x ; hence

by these substitutions the foregoing series become

$$\cos. x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.$$

$$\sin. x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

by means of which we may calculate the values of the sine and cosine of any arc x , in parts of the radius or linear unit, when we know the length of x itself, according to the same scale. The length of any arc in parts of the radius is easily ascertained from the known value of 180° , or of a semicircle, in those parts, which, by putting π for the semicircumference to radius 1, is (see Geom. p. 133)

$$\pi = 3.14159265358979, \&c.$$

so that the length of an arc of x degrees is $\frac{x}{180} \cdot \pi = \frac{x}{90} \cdot \frac{\pi}{2}$.

As in calculating the sines and cosines x may be always taken less than

90, it follows that $\frac{x}{90}$ will be a decimal fraction; if we call this m

we may write the foregoing series thus,

$$\sin. (m \cdot 90^\circ) = m \frac{\pi}{2} - \frac{(\frac{1}{2} \pi)^3}{1 \cdot 2 \cdot 3} m^3 + \frac{(\frac{1}{2} \pi)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} m^5 - \&c.$$

$$\cos. (m \cdot 90^\circ) = 1 - \frac{(\frac{1}{2} \pi)^2}{1 \cdot 2} m^2 + \frac{(\frac{1}{2} \pi)^4}{1 \cdot 2 \cdot 3 \cdot 4} m^4 - \&c.$$

which series are now in a form suited to immediate calculation.

Suppose, for example, the sine and cosine of $1'$ are required, then

$$m = \frac{1}{90 \times 60} \therefore m \frac{\pi}{2} = .0002908882, \&c.$$

$$\therefore \sin. 1' = .0002908882, \&c. - \frac{1}{1 \cdot 2 \cdot 3} (.0002908882, \&c.)^3 + \&c. = .0002908882, \&c.$$

$$\cos. 1' = 1 - \frac{1}{1 \cdot 2} (.0002908882, \&c.)^2 + \&c. =$$

$$.9999999577 \&c.$$

and from knowing the value of $\sin. 1'$ and $\cos. 1'$ we might compute the sines and cosines for every minute in the quadrant, by means of the formula (3), which, when $B = 1'$, becomes

$$\sin. (A + 1') = 2 \sin. A \cos. 1' - \sin. (A - 1'),$$

in which A is to be made successively equal to $1', 2', 3', \&c.$ But we shall not enter into the details of this computation here, our present object being to deduce formulas for the sines, cosines, &c. of multiple arcs.

From the general expressions already given for $\sin. nA$, and $\cos. nA$, those for $\tan. nA$, $\cot. nA$, &c. may be readily obtained by help of the equations at (9); we shall not, therefore, occupy the space by writing them down, but confine ourselves throughout the remainder of this article entirely to the consideration of double arcs, as formulas for these are in much more frequent request than for any higher multiple. The formulas of which we speak may, of course, all be deduced from the general expressions investigated in the beginning of this article, but, for simplicity sake, we shall go nearer to first principles, and deduce them from the expressions in art. (26).

Referring to the equations (1), (2), art. (26), we have when $A = B$,

$$\sin. 2A = 2 \sin. A \cos. A \dots (18)$$

$$\cos. 2A = \cos.^2 A - \sin.^2 A, \text{ or } \cos. 2A = 2 \cos.^2 A - 1,$$

$$\text{or } \cos. 2A = 1 - 2 \sin.^2 A \dots (19);$$

and from the last two of these we immediately get

$$\cos. A = \sqrt{\frac{1}{2} + \frac{1}{2} \cos. 2A}, \sin. A = \sqrt{\frac{1}{2} - \frac{1}{2} \cos. 2A} \dots (20);$$

and, therefore, by division,

$$\tan. A = \sqrt{\frac{1 - \cos. 2A}{1 + \cos. 2A}}, \cot. A = \sqrt{\frac{1 + \cos. 2A}{1 - \cos. 2A}} \dots (21);$$

from which we get two new expressions for $\cos. 2A$, viz.

$$\cos. 2A = \frac{1 - \tan.^2 A}{1 + \tan.^2 A} = \frac{\cot.^2 A - 1}{\cot.^2 A + 1} \dots (22).$$

If instead of A we write $45^\circ - A$, then since $\cos. (90^\circ - 2A) = \sin. 2A$, we have

$$\sin. 2A = \frac{1 - \tan.^2 (45^\circ - A)}{1 + \tan.^2 (45^\circ - A)} = \frac{\cot.^2 (45^\circ - A) - 1}{\cot.^2 (45^\circ - A) + 1}.$$

It may be worth while to remark that the radical in the above ex-

pressions for $\tan. A$, $\cot. A$, may be removed by multiplying the numerator and denominator of each fraction by its numerator: we thus have

$$\tan. A = \frac{1 - \cos. 2 A}{\sin. 2 A}, \quad \cot. A = \frac{1 + \cos. 2 A}{\sin. 2 A}.$$

For the tangent and cotangent of a double arc we have, by division, (18), (19),

$$\frac{\sin. 2 A}{\cos. 2 A} = \frac{2 \sin. A \cos. A}{\cos.^2 A - \sin.^2 A};$$

that is, dividing numerator and denominator of the second member by $\cos.^2 A$, or by $\sin.^2 A$, and recollecting that $\frac{\sin.}{\cos.} = \tan.$, and that $\frac{1}{\tan.} = \cot.$, we have

$$\left. \begin{aligned} \tan. 2 A &= \frac{2 \tan. A}{1 - \tan.^2 A} = \frac{2 \cot. A}{\cot.^2 A - 1} = \frac{2}{\cot. A - \tan. A} \\ \cot. 2 A &= \frac{1 - \tan.^2 A}{2 \tan. A} = \frac{\cot.^2 A - 1}{2 \cot. A} = \frac{1}{2} (\cot. A - \tan. A); \end{aligned} \right\} (23)$$

which expressions also immediately come from the values of $\tan. (A + B)$, $\cot. (A + B)$, at (26), by putting $A = B$.

Comparing the above value of $\tan. 2 A$ with the expression (8), art. (26), we have,

$$2 \tan. 2 A = \tan. (45^\circ + A) - \tan. (45^\circ - A);$$

or which is the same thing,

$$2 \tan. A = \tan. (45^\circ + \frac{1}{2} A) - \tan. (45^\circ - \frac{1}{2} A) \dots (V).$$

Formulas for the secants and cosecants of double arcs are easily deduced from those for the cosine and sine, because $\sec. = \frac{1}{\cos.}$, and

$\csc. = \frac{1}{\sin.}$, thus, from equation (22) above, we have

$$\sec. 2 A = \frac{1 + \tan.^2 A}{1 - \tan.^2 A} = \frac{\sec.^2 A}{2 - \sec.^2 A};$$

and, from equation (18),

$$\operatorname{cosec}. 2 A = \frac{1}{2 \sin. A \cos. A} = \frac{1}{2} \sec. A \operatorname{cosec}. A.$$

(31.) Another useful class of formulas are those for half arcs; they may be easily deduced from the expressions for the double arcs: thus putting $\frac{1}{2} A$ for A , we have from (20).

$$\sin. \frac{1}{2} A = \sqrt{\frac{1}{2} - \frac{1}{2} \cos. A}, \quad \cos. \frac{1}{2} A = \sqrt{\frac{1}{2} + \frac{1}{2} \cos. A} \dots (24);$$

also from (21),

$$\left. \begin{aligned} \tan. \frac{1}{2} A &= \sqrt{\frac{1 - \cos. A}{1 + \cos. A}} = \frac{1 - \cos. A}{\sin. A} \\ \cot. \frac{1}{2} A &= \sqrt{\frac{1 + \cos. A}{1 - \cos. A}} = \frac{1 + \cos. A}{\sin. A} \end{aligned} \right\} \dots (25).$$

Other useful values of $\sin. \frac{1}{2} A$, and $\cos. \frac{1}{2} A$, are derivable from the equation (18) last article, for when $\frac{1}{2} A$ is put for A the equation is

$$\sin. A = 2 \sin. \frac{1}{2} A \cos. \frac{1}{2} A \dots (26),$$

and if this be either added to or subtracted from

$$1 = \sin.^2 \frac{1}{2} A + \cos.^2 \frac{1}{2} A,$$

the second member will become in each case a perfect square, viz.

$$1 + \sin. A = (\sin. \frac{1}{2} A + \cos. \frac{1}{2} A)^2$$

$$1 - \sin. A = (\sin. \frac{1}{2} A - \cos. \frac{1}{2} A)^2;$$

hence,

$$\sqrt{1 + \sin. A} = \sin. \frac{1}{2} A + \cos. \frac{1}{2} A$$

$$\sqrt{1 - \sin. A} = \sin. \frac{1}{2} A - \cos. \frac{1}{2} A.$$

Let A be less than 90° , then the radical must be taken positive in the first, and negative in the second expression; hence, by addition and subtraction,

$$\sin. \frac{1}{2} A = \frac{1}{2} \{ \sqrt{1 + \sin. A} - \sqrt{1 - \sin. A} \} \dots (V).$$

$$\cos. \frac{1}{2} A = \frac{1}{2} \{ \sqrt{1 + \sin. A} + \sqrt{1 - \sin. A} \}$$

By means of these two expressions the accuracy of a table of sines

and cosines may be examined; that is to say, from the calculated values $\sin. A$, in the table, we may compute, by these equations, the values of $\sin. \frac{1}{2} A$, and of $\cos. \frac{1}{2} A$; if these agree with the tabular values, found by other means, we may conclude that the tables are correct in the parts thus verified. Formulas employed in this manner to put the accuracy of the tables to the test are called *formulas of verification*. We have given three of these, and marked them with the letter (V).

(32.) The following formulas involving the half sums and half differences of two arcs are of frequent application: substitute $\frac{1}{2}(A + B)$ for A , and $\frac{1}{2}(A - B)$ for B , in the equations (3), (4), at art. (26) and we have

$$\left. \begin{aligned} \sin. A + \sin. B &= 2 \sin. \frac{1}{2}(A + B) \cos. \frac{1}{2}(A - B) \\ \cos. A + \cos. B &= 2 \cos. \frac{1}{2}(A + B) \cos. \frac{1}{2}(A - B) \\ \sin. A - \sin. B &= 2 \cos. \frac{1}{2}(A + B) \sin. \frac{1}{2}(A - B) \\ \cos. B - \cos. A &= 2 \sin. \frac{1}{2}(A + B) \sin. \frac{1}{2}(A - B) \end{aligned} \right\} \dots (27);$$

and from these we get, by division,

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \frac{1}{2}(A + B); \quad \frac{\sin. A - \sin. B}{\cos. B - \cos. A} = \cot. \frac{1}{2}(A + B). \quad (28)$$

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \tan. \frac{1}{2}(A - B); \quad \frac{\sin. A + \sin. B}{\cos. B - \cos. A} = \cot. \frac{1}{2}(A - B). \quad (29)$$

In each of these expressions let $A = 90^\circ$, and we shall have

$$1 + \sin. B = 2 \sin. (45^\circ + \frac{1}{2} B) \cos. (45^\circ - \frac{1}{2} B) = 2 \sin.^2 (45^\circ + \frac{1}{2} B)$$

$$\cos. B = 2 \cos. (45^\circ + \frac{1}{2} B) \cos. (45^\circ - \frac{1}{2} B) = 2 \cos.^2 \frac{1}{2} B - 1, \text{ by eq. 18,}$$

$$\begin{aligned} 1 - \sin. B &= 2 \cos. (45^\circ + \frac{1}{2} B) \sin. (45^\circ - \frac{1}{2} B) = 2 \cos.^2 (45^\circ + \frac{1}{2} B) \\ &= 2 \sin.^2 (45^\circ - \frac{1}{2} B) \end{aligned}$$

$$1 - \cos. B = 2 \sin. (45^\circ + \frac{1}{2} B) \sin. (45^\circ - \frac{1}{2} B) = 2 \sin.^2 \frac{1}{2} B, \text{ by eq. 19,}$$

$$\begin{aligned} \frac{1 + \sin. B}{\cos. B} &= \tan. (45^\circ + \frac{1}{2} B), \quad \frac{1 - \sin. B}{\cos. B} = \cot. (45^\circ + \frac{1}{2} B) \\ &= \cot. (45^\circ - \frac{1}{2} B), \quad = \tan. (45^\circ - \frac{1}{2} B) \end{aligned}$$

$$\frac{1 + \sin. B}{1 - \sin. B} = \tan.^2 (45^\circ + \frac{1}{2} B), \quad \frac{1 + \cos. B}{1 - \cos. B} = \cot.^2 \frac{1}{2} B.$$

Again, dividing (28) by (29), we have

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}; \quad \frac{\sin. A - \sin. B}{\sin. A + \sin. B} = \frac{\cot. \frac{1}{2}(A+B)}{\cot. \frac{1}{2}(A-B)} \dots (30)$$

Lastly, substituting $A+B$ for A in (26) last article, we have

$$\sin. (A+B) = 2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A+B);$$

and dividing this by each of the formulas (27) in succession, there results

$$\frac{\sin. (A+B)}{\sin. A + \sin. B} = \frac{\cos. \frac{1}{2}(A+B)}{\cos. \frac{1}{2}(A-B)}$$

$$\frac{\sin. (A+B)}{\cos. A + \cos. B} = \frac{\sin. \frac{1}{2}(A+B)}{\cos. \frac{1}{2}(A-B)}$$

$$\frac{\sin. (A+B)}{\sin. A - \sin. B} = \frac{\sin. \frac{1}{2}(A+B)}{\sin. \frac{1}{2}(A-B)}$$

$$\frac{\sin. (A+B)}{\cos. B - \cos. A} = \frac{\cos. \frac{1}{2}(A+B)}{\sin. \frac{1}{2}(A-B)}.$$

(33.) We shall conclude this chapter on the theory of the trigonometrical lines, with two curious and useful propositions.

1. *To express the sine and cosine of a real arc by means of imaginary exponentials.*

By the *exponential theorem*,*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \&c.$$

where e represents the base of the Naperian logarithms, that is, $e = 2.7182818, \&c.$

For x substitute $x\sqrt{-1}$, and $-x\sqrt{-1}$ successively, and we have these developments,

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \&c. \quad (1)$$

* See the "Elementary Essay on the construction of Logarithms," page 68.

$$e^{x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{2} + \frac{x^3\sqrt{-1}}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} - \&c. \quad (2);$$

hence, by addition,

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2 \left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} - \&c. \right)$$

But by art. (30) the series on the right is the development of $\cos. x$, hence

$$\cos. x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

By subtracting (2) from (1) we have

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1} \left(x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c. \right)$$

But by art. (30) the series on the right is the development of $\sin. x$; hence

$$\sin. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

2. To develop $\sin.^n x$, $\cos.^n x$, in terms of the sine and cosine of the multiples of x .

Put

$$\left. \begin{aligned} \cos. x + \sin. x \cdot \sqrt{-1} &= u \\ \cos. x - \sin. x \cdot \sqrt{-1} &= v \end{aligned} \right\} \dots (1),$$

then art. (29),

$$\left. \begin{aligned} \cos. nx + \sin. nx \cdot \sqrt{-1} &= u^n \\ \cos. nx - \sin. nx \cdot \sqrt{-1} &= v^n \end{aligned} \right\} \dots (2);$$

from which, by addition and multiplication, we get

$$u^n + v^n = 2 \cos. nx, \quad u^n v^n = 1 \dots (3).$$

Add together the equations (1); there will result

$$\cos. x = \frac{1}{2} (u + v);$$

and, therefore,

$$\cos.^n x = \frac{1}{2^n} (u + v)^n = \frac{1}{2^n} (v + u)^n;$$

hence, by the binomial theorem,

$$\cos.^n x = \frac{1}{2^n} \{ u^n + nu^{n-1}v + \frac{n(n-1)}{2} u^{n-2}v^2 + \&c. \},$$

or,

$$\cos.^n x = \frac{1}{2^n} \{ v^n + nv^{n-1}u + \frac{n(n-1)}{2} v^{n-2}u^2 + \&c. \};$$

adding these equations together, and dividing by 2, we have

$$\begin{aligned} \cos.^n x = \frac{1}{2^n + 1} \{ & u^n + v^n + nuv(u^{n-2} + v^{n-2}) + \\ & \frac{n(n-1)}{2} u^2 v^2 (u^{n-4} + v^{n-4}) + \&c. \}. \end{aligned}$$

But from (3)

$u^n + v^n = 2 \cos. nx$	$uv = 1$
$u^{n-2} + v^{n-2} = 2 \cos. (n-2)x$	$u^2 v^2 = 1$
$u^{n-4} + v^{n-4} = 2 \cos. (n-4)x$	$u^4 v^4 = 1$
$\&c. \qquad \&c.$	$\&c. \quad \&c.$

hence, by substitution, the development of $\cos.^n x$ becomes

$$\begin{aligned} \cos.^n x = \frac{1}{2^n} \{ & \cos. nx + n \cos. (n-2)x + \\ & \frac{n(n-1)}{2} \cos. (n-4)x + \&c. \} \dots (4). \end{aligned}$$

Again, subtract the second of (1) from the first, and we have

$$2 \sin. x \cdot \sqrt{-1} = u - v \therefore \sin. x = \frac{u - v}{2 \sqrt{-1}};$$

and, consequently,

$$\sin.^n x = \frac{(u - v)^n}{(2 \sqrt{-1})^n}.$$

1. Let n be even, then (Algebra, p. 149,)

$$(u-v)^n = (v-u)^n;$$

hence,

$$\sin.^n x = \frac{1}{(2\sqrt{-1})^n} (u-v)^n, \text{ or } \sin.^n x = \frac{1}{(2\sqrt{-1})^n} (v-u)^n;$$

and by adding these equations together after having developed $(u-v)^n$, and $(v-u)^n$, we have

$$2 \sin.^n x = \frac{1}{(2\sqrt{-1})^n} \{u^n + v^n - nuv(u^{n-2} + v^{n-2}) + \frac{n(n-1)}{2} u^2 v^2 (u^{n-4} + v^{n-4}) - \&c.\};$$

and making the same substitutions as before in virtue of (3), and recollecting that, because n is even, $(\sqrt{-1})^n = \mp 1$, the upper sign having place when n is either of the numbers 2, 6, 10, &c. and the lower sign when n is either of the numbers 4, 8, 12, &c. we have for the development of $\sin.^n x$

$$\sin.^n x = \mp \frac{1}{2^n} \{ \cos. nx - n \cos. (n-2)x + \frac{n(n-1)}{2} \cos. (n-4)x - \&c. \} \dots (5).$$

2. Let n be odd, then

$$(u-v)^n = (-1)^n (v-u)^n = -(v-u)^n;$$

therefore,

$$\sin.^n x = \frac{1}{(2\sqrt{-1})^n} (u-v)^n, \text{ or } \sin.^n x = -\frac{1}{(2\sqrt{-1})^n} (v-u)^n;$$

and developing $(u-v)^n$, $(v-u)^n$ as before, and taking the sum of these equations, we have

$$2 \sin.^n x = \frac{1}{(2\sqrt{-1})^n} \{u^n - v^n - nuv(u^{n-2} - v^{n-2}) + \frac{n(n-1)}{2} u^2 v^2 (u^{n-4} - v^{n-4}) - \&c.\}$$

But from the equations (2),

$$u^n - v^n = 2 \sin. nx \sqrt{-1}, \quad u^n v^n = 1;$$

consequently, since $(\sqrt{-1})^{n-1} = \mp 1$, the foregoing development becomes

$$\sin.^n x = \mp \frac{1}{2^n} \{ \sin. nx - n \sin. (n-2)x +$$

$$\frac{n(n-1)}{2} \sin. (n-4)x - \&c. \};$$

the upper sign having place when $n-1$ is either of the numbers 2, 6, 10, &c. and the lower sign having place when $n-1$ is either of the numbers 4, 8, 12, &c. The general term of the first series of numbers is $4m+2$, that of the second series $4m$.

END OF PART I.

Let from the equation (2),

$$u = \frac{1}{2} \left(\sqrt{1 + \frac{4}{a^2}} + \sqrt{1 - \frac{4}{a^2}} \right), \quad \text{or } u = 1;$$
 consequently, since $\sqrt{1 - \frac{4}{a^2}} = \pm 1$, the foregoing development becomes

$$u^2 = \frac{1}{2} \left(\sqrt{1 + \frac{4}{a^2}} + \sqrt{1 - \frac{4}{a^2}} \right)^2 = 1 + \frac{2}{a^2} \pm \frac{4}{a^2}.$$

$$\frac{u(a-1)}{2} \pm \frac{u(a+1)}{2} = (a-1) \pm (a+1).$$

the upper sign having place when $a-1$ is either of the numbers 2, 6, 10, &c. and the lower sign having place when $a-1$ is either of the numbers 4, 8, 12, &c. The general term of the first series of numbers is $a-1$, that of the second series $2a$.

END OF PART I.

PART II.

ELEMENTS OF SPHERICAL TRIGONOMETRY.

CHAPTER I.

ON THE SPHERE.

(34.) A SPHERE is a solid whose surface is every where equally distant from a certain point within it, called the centre. It may be generated by the revolution of a semicircle about the diameter.

Any line drawn from the centre to the surface of the sphere is called the radius; and the line through the centre having both its extremities in the surface is the diameter.

A plane surface, or simply a plane, is that in which if any two points whatever be taken, the straight line which joins them shall lie wholly in that surface.

A plane may be drawn through any three points, taken at random in space, but not through more than three; for having joined two of the proposed points by a straight line we may pass a plane through this line in any direction, and we may turn it round upon this line till it arrives at the other point. Three points, therefore, not in the same straight line, fix the position of a plane.

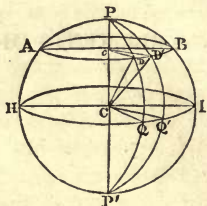
It follows from this that the common intersection of two planes must be a straight line; for, if among the points in the intersection there be three which are not in the same straight line, the two planes passing through them must coincide and form but one.

A straight line is said to be perpendicular to a plane when it is perpendicular to every straight line in that plane, drawn through its *foot*,

or the point where the perpendicular meets the plane. These definitions will suffice for the purpose of establishing the necessary preliminary theorems of spherical Geometry.

(35.) *If a sphere be any how cut by a plane, the section must be a circle.*

Let C be the centre of the sphere, and ADB the plane section; draw Cc perpendicular to this plane, and from c draw any line cD in the section and terminating at the surface; then the angle CcD must be a right angle. Join CD , then wherever the point D may be, CD will always be of the same constant length, being the radius of the sphere; and in conse-



quence of the right angle c , $cD = \sqrt{CD^2 - Cc^2}$; hence cD must have the same constant length in whatever direction it be drawn, that is, the bounding line ADB is the circumference of a circle of which c is the centre.

The circle is, obviously, the larger, as it is nearer to the centre C of the sphere, or as its perpendicular distance Cc is less, because CD being constant cD increases as Cc diminishes, and becomes the greatest possible when Cc is 0, that is, when the section passes through the centre of the sphere; hence every circle whose plane passes through the centre of the sphere is called a *great circle* of the sphere, and every other a *small circle*.

It is obvious that the circumference of a great circle may be drawn through any two points on the surface of a sphere, because a plane may be drawn through these two points and through the centre also, but a great circle cannot be drawn through three points on the surface, taken at random, because then a plane might be drawn through four points taken at random; a circle of some kind, however, may always be drawn through three points on the surface of the sphere, since a plane may be drawn through them.

The line Cc from the centre of the sphere perpendicular to the plane of the circle passes, as we have seen, through its centre c ; if this line be produced both ways to the surface of the sphere, the opposite points P , P' , are called the *poles* of the circle. Thus every circle on the sphere has two poles diametrically opposite, the diameter which joins them

being perpendicular to the plane of the circle. The poles of a small circle are unequally distant from its plane, the inequality of distance amounting to twice Cc ; but in a great circle this inequality vanishes, and the poles are equidistant from the circle.

As the poles of any circle are at the extremities of a diameter of the sphere, an infinite number of great circles may be drawn through them; indeed, every circle passing through them will necessarily be a great circle, because the entire diameter joining them must be comprised in every plane drawn through them. The distance of any circle from either of its poles, measured upon any of these infinite number of great circles, is constantly the same, that is, the distances or arcs PB , PD' , PD , PA , &c. are equal, because the constant line Pc is the common versed sine of all these arcs to the common radius CP ; hence the other distances $P'B$, $P'E$, &c. must be equal. Every arc of a great circle is thus distant from either pole by a quadrant or 90° .

(36.) *Two great circles always intersect in two points, at the distance of a semicircle from each other, that is, the circumferences bisect each other.* For as the plane of each circle passes through the centre of the sphere their intersection must be a diameter common to both circles, and it is at the extremities of this diameter that the circumferences cross each other.

From this we learn that if from any point on the sphere two quadrantal arcs can be drawn to two points in any great circle, the distance between the points being less than 180° , then the first point must be the pole of this great circle; for it is necessarily the pole of some great circle passing through the proposed points, and as only one great circle can pass through two points, which are not 180° apart, the pole must belong to the circle in question.

In spherical trigonometry, the arcs of great circles only are concerned, and the angle included between two such arcs, that is to say, a spherical angle, is measured in a manner analogous to that in which a plane angle is measured. For the measure of a plane angle we take the intercepted arc of that circle whose centre is at the vertex, and whose radius is some assumed unit: in like manner for the measure of a spherical angle we take the intercepted arc of that circle whose pole is at the vertex, and whose radius is some fixed unit, viz. the radius of the sphere on whose surface the angle is: thus, in the foregoing figure the

spherical angle DPD' is measured by the intercepted arc QQ' , of which the pole is P , and radius, CQ , that of the sphere.

It is as easy to justify the propriety of adopting this mode of measuring spherical angles as it is to justify the method of measuring plane angles, for in both cases the intercepted arc varies as the angle; this, by the way, is true of the intercepted arc DD' of any small circle whose pole is P , but we are compelled to refer the measure to a *great* circle, in order that all the trigonometrical lines concerned in the same inquiry may be related to a common radius, for, as we have before remarked, the sides of a spherical triangle are always arcs of great circles.

From what we have just said it appears that a spherical angle DPD' has the same measure as either of the equal plane angles QCQ' , DcD' , &c. situated in the planes of the circles whose common pole is P , and whose sides are formed by the intersection of these planes with those of the two great circles, forming the sides of the spherical angle. If at P tangents were drawn to the two great circles PD , PD' , and in their planes they would obviously include the same angle as the lines CQ , CQ' , to which they are parallel; indeed if we conceive the plane of the circle HQQ' to move parallel to itself towards the pole, P , the path of C being along the line CP , the angle QCQ' will successively coincide with QCQ' , DcD' , &c. till C coincides with P , when the lines CQ , CQ' , will become tangents to the circles at P , and will remain each in the plane along which it has moved; hence *the measure of the angle included between these tangents is also the measure of the spherical angle.*

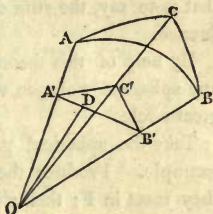
(37.) If in the plane of HQI perpendiculars be drawn from C to each of the planes of the circles PQP' , $PQ'P'$, these will be perpendicular to the lines CQ , CQ' , and will therefore, include the same angle, which angle will be measured by the arc of HQI , which the said perpendiculars intercept; but these perpendiculars will meet the surface at the poles of the circles to whose planes they are perpendicular; hence *the great circle distance between the poles of two intersecting great circles measures their angle of intersection.*

Every great circle which passes through the poles of another is at right angles to it. Thus the great circle $PDQP'$, through the poles of $HQQ'I$, is at right angles to $HQQ'I$; for if a tangent were drawn to PQP' at the point Q it would be in the same plane with and parallel to CP , and

if a tangent were drawn to HQI at the point Q it would be in the same plane with and parallel to CH; hence if these two tangents were to move simultaneously parallel to themselves, the path of their point of concurrence Q being along QC, they would necessarily coincide with the perpendiculars CP, CH, when Q arrived at C: these tangents, therefore, form a right angle; hence the great circles are perpendicular to each other, or the spherical angle at Q is a right angle.

(38.) *Any one side of a spherical triangle is less than the sum of the other two.*

Let ABC be any spherical triangle, and O the centre of the sphere; draw the radii OA, OB, OC, then there will be about O three angles in three distinct planes respectively, measured by the arcs AB, BC, CA. Let AB be the greatest of these arcs, then it will only be necessary to show that $AB < AC + CB$, or that $AOB < AOC + BOC$. In the plane of AOB draw any line A'B', and then draw OD, making an angle B'OD equal to BOC; make OC' equal to OD, and join CB', C'A'.



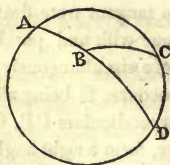
Then since by construction the two sides B'O, OD, and the included angle, are respectively equal to the two sides B'O, OC', and the included angle, $B'D = B'C'$. But in the plane triangle A'B'C', $A'B' < A'C' + B'C'$ $\therefore A'D < A'C'$; hence the two sides OA', OD, of the triangle A'OD, are equal to the two sides OA', OC', of the triangle A'OC', but the third side A'D of the former is less than the third side A'C' of the latter, and, consequently, $A'OD < A'OC'$; hence, since B'OD has been made equal to B'OC', it follows that

$$A'OD + B'OD = A'OB' < A'OC' + B'OC'$$

$$\therefore AB < AC + CB.$$

(39.) *The sum of all the three sides of a spherical triangle is less than the circumference of a great circle.*

Let ABC be any spherical triangle; produce the sides AB , AC , till they meet again in D , then the arcs ABD , ACD , will be semi-circumferences, since (36,) two great circles always bisect each other. But in the triangle BCD we have $BC < BD + CD$, and, consequently, by adding $AB + AC$ to both, we shall have

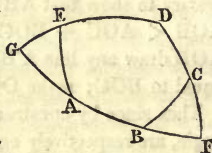


$$AB + AC + BC < ABD + ACD;$$

that is to say, the sum of the three sides is less than a whole circumference.

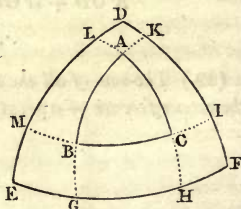
By help of this theorem we may show that the sum of the sides of any spherical polygon whatever is less than the circumference of a great circle.

Take the spherical pentagon $ABCDE$ for example. Produce the sides AB , DC , till they meet in F ; then since $BC < BF + CF$, the perimeter of the pentagon will be less than the quadrilateral $AEDF$. Again, produce the sides DE , BA , till they meet in G ; we shall have $EA < EG + AG$; hence the perimeter of the quadrilateral $AEDF$ is less than that of the triangle DFG ; which last is itself less than the circumference of a great circle; the perimeter of the original polygon is, therefore, less still.



(40.) *If from the three vertices of a spherical triangle, taken as poles, arcs be described, forming a new triangle, then the vertices of the new triangle will be the poles of the other triangle.*

For let ABC be any spherical triangle, and with the pole A , and circular radius AG equal to a quadrant, describe the arc EF ; in like manner with the pole B and same radius describe the arc FD , meeting the former in F ; and, lastly, with the pole C and same radius describe the arc ED , completing the spherical triangle DEF .



Then, because the arcs, whose poles are A and C , intersect at E , the

points A, C, are each 90° distant from E; and as the arc AC is less than 180° , E must be the pole of AC (36). In like manner is it shown that F is the pole of AB, and D the pole of BC.

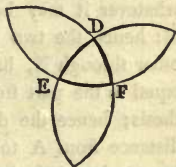
The triangle DEF is sometimes, from the mode of its construction, called the *polar triangle*, and the original one ABC the *primitive triangle*.

(41.) *Any angle of the primitive triangle is the supplement of the side opposite to it of the polar triangle, and any angle of the polar triangle is the supplement of the side opposite to it in the primitive triangle.*

For EH being the radius of HL is $= 90^\circ$, and FG being the radius of GK is also $= 90^\circ$, and the sum of these radii, namely, $EF + GH = 180^\circ$, therefore, GH, which is the measure of the angle A, is the supplement of the side EF opposite to it. In like manner is it shown that B is the supplement of DF, and C the supplement of DE. Again, BI being the radius of ID, and CM the radius of MD, the sum of these $MI + BC = 180^\circ$; therefore, BC is the supplement of MI, which measures the angle D.

On account of the property just demonstrated, the triangles ABC, DEF, are frequently called *supplemental triangles*.

It is proper to remark here, as *Legendre* has done, that besides the triangle DEF three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the others by the circumstance that the two angles A and D (see preceding fig.) be on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.



(42.) From the foregoing proposition it follows that *the three angles of every spherical triangle are together greater than two right angles, and less than six.*

For the sides of the supplemental triangle DEF are together less than four right angles (39), and as these are supplements of the angles A, B, C, and therefore when added to them make six right angles, these last must together exceed two right angles. But they cannot amount to six right angles, for in that case the sum of the sides of the supplemental triangle would be 0, which is absurd. Hence, unlike plane triangles,

a spherical triangle may have all its angles right angles or all obtuse angles.

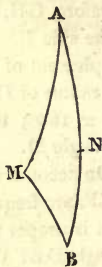
(43.) The foregoing geometrical properties comprise all that we require, for the foundation of the analytical theory of spherical Trigonometry: we need not, therefore, enumerate any more. We shall, however, in conclusion, endeavour to establish the fact that *the arc of a great circle joining two points is the shortest line that can be drawn on the sphere from the one to the other.*

The following proof of this property is by *Legendre*.

Let ANB be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point in the shortest path, between A and B. Through the point M draw MA, MB, arcs of great circles; and take $BN = MB$.

Then, by (38), the arc ANB is shorter than $AM + MB$; take $BN = BM$, respectively from both; there will remain $AN < AM$.

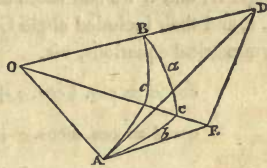
Now, the distance of B from M, whether it be the same with the arc BM or with any other line, is equal to the distance of B from N; for by making the plane of the great circle BM revolve about the diameter, which passes through B, the point M may be brought into the position of the point N; and the shortest line between M and B, whatever it may be, will then be identical with that between N and B: hence the two paths from A to B, one passing through M, the other through N, have an equal part in each, the part from M to B equal to the part from N to B. The first path is the shorter by hypothesis; hence the distance from A to M must be shorter than the distance from A to N; which is absurd, the arc AM being proved greater than AN; hence no point of the shortest line from A to B can be out of the arc ANB; hence this arc is itself the shortest distance between its two extremities.



CHAPTER II.

INVESTIGATION OF FORMULAS, AND RULES FOR THE SOLUTION OF SPHERICAL TRIANGLES.

(44.) Let ABC be a triangle traced on the surface of a sphere of which the centre is O, and the radius equal to the linear unit. The angles of this triangle we shall represent by the letters at their vertices, A, B, C, and the sides opposite to them by the small letters a, b, c ; so that having drawn the two tangents AD, AE, to meet the radii OB, OC, produced through the other extremities of the arcs AB, AC, we shall have



$$\left. \begin{aligned} AD &= \tan. c = \frac{\sin. c}{\cos. c}, & AE &= \tan. b = \frac{\sin. b}{\cos. b} \\ OD &= \sec. c = \frac{1}{\cos. c}, & OE &= \sec. b = \frac{1}{\cos. b} \end{aligned} \right\} \dots (1).$$

Draw DE, then in the two triangles ODE, ADE, we have (17)

$$DE^2 = OE^2 + OD^2 - 2 OE \cdot OD \cos. a$$

$$DE^2 = AE^2 + AD^2 - 2 AE \cdot AD \cos. A;$$

recollecting that (p. 68) the plane angle DAE measures the spherical angle A.

Substituting in these equations the values given by (1), they become

$$DE^2 = \sec.^2 b + \sec.^2 c - \frac{2 \cos. a}{\cos. b \cos. c}$$

$$DE^2 = \tan.^2 b + \tan.^2 c - \frac{2 \sin. b \sin. c \cos. A}{\cos. b \cos. c};$$

$\therefore$ by subtraction

$$0 = 1 + 1 + (\sin. b \sin. c \cos. A - \cos. a) \frac{2}{\cos. b \cos. c}.$$

Hence multiplying by $\frac{\cos. b \cos. c}{2}$, and transposing, we have

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A;$$

which is a general expression for the cosine of any side in terms of the other two sides, and their included angle. If we had taken the side b instead of a , the other two would have been a, c , and their included angle B ; and if we had taken the side c the other two would have been a, b , and their included angle C ; we have, therefore, the three following symmetrical equations, viz.

$$\left. \begin{aligned} \cos. a &= \cos. b \cos. c + \sin. b \sin. c \cos. A \\ \cos. b &= \cos. a \cos. c + \sin. a \sin. c \cos. B \\ \cos. c &= \cos. a \cos. b + \sin. a \sin. b \cos. C \end{aligned} \right\} \dots (A);$$

and these equations embody the whole theory of spherical trigonometry and are sufficient to supply rules for the solution of every case.

(45.) Some interesting geometrical properties flow also from these equations.

1. Suppose two sides b, c , of the triangle are equal, that is, let it be isosceles, then it will follow from the two last of these equations that, like as in the isosceles plane triangle, *the angles opposite the equal sides will be equal*. For taking the difference of these two equations on the supposition that $b = c$; we have

$$0 = \sin. a \sin. b \cos. B - \sin. a \sin. b \cos. C;$$

and, consequently, $B = C$.

2. If $a = b = c$, then it is in a similar manner proved that $A = B = C$, that is, *every equilateral spherical triangle is equiangular*.

3. *The arc which bisects the vertical angle A of a spherical isosceles triangle also bisects the base a.* For let p represent this bisecting arc, and m, m' , the parts into which it divides the base, then the two spherical triangles thus formed give, by the above equations,

$$\cos. m = \cos. b \cos. p + \sin. b \sin. p \cos. \frac{1}{2} A$$

$$\cos. m' = \cos. a \cos. p + \sin. a \sin. p \cos. \frac{1}{2} A;$$

therefore, since by hypothesis $a = b$ we have $m = m'$, that is, the arc

bisecting the vertical angle also bisects the base, and the student will find no difficulty in further showing that this same arc is also perpendicular to the base.

4. *If two sides and the included angle in one triangle are equal to two sides, and the included angle in another, the third side of the one must be equal to the third side of the other.* This is obvious from the first of (A), which shows that $\cos. a$, and therefore a , becomes fixed when the other two sides b, c , and their included A , is fixed; moreover, the remaining angles of the one triangle are equal to the remaining angles of the other; for by the second and third of (A), $\cos. B, \cos. C$, and, therefore, B, C , become fixed when a, b , and c , are fixed.

5. *If the three sides of one triangle are severally equal to the three sides of another, the three angles of the one are also severally equal to those of the other, the equal angles being opposite to the equal sides.* For with fixed values for a, b, c , the formulas (A) give fixed values for $\cos. A, \cos. B, \cos. C$, and, therefore, for A, B, C . We may, in like manner, infer the equality of the sides from that of the angles, but perhaps the inference is a little more obvious from the equations (B), p. 81, following.

In these deductions the student will observe that we have abstained from saying that the triangles are *equal in all respects* as in the analogous theorems of plane geometry; because two spherical triangles may exist, of which the several *parts* of the one may be equal to the several parts of the other, and yet not admit of coincidence, as plane triangles would under like conditions. Thus, if two plane triangles, of which the sides in the one are equal to those in the other, be joined together by a corresponding side of each, and if we turn one of the triangles about this common side either above or below the plane on which they are situated till it comes to that plane again, we know that we shall thus obtain a perfect coincidence between the two; but if the sides of the triangles thus joined are the chords of two spherical triangles, these triangles will, as we have seen, have all their parts equal, each to each, because, the chords being equal, the arcs must be equal, and yet it is very plain that the corresponding parts of the two triangles cannot be brought into coincidence as in plane triangles, and only in the particular case in which the two triangles are isosceles can they coincide, by being laid the one over the other. We cannot therefore say, as in plane triangles,

that two triangles, whose corresponding parts are equal, have equal surfaces, without distinct proof. This proof will be given in Part iv.

We shall add here but one more inference from the fundamental equations (A).

6. By the first of (A) if the sides b, c , are fixed, $\cos. a$ will necessarily diminish as $\cos. A$ diminishes; that is, a will increase as A increases: hence *if two triangles have two sides in the one equal to two sides in the other, but the included angle in the first greater than the included angle in the second, then the third side of the first triangle must be greater than the third side of the second.*

Let us now proceed with the analytical discussion.

The three general equations above involve all the six parts of a triangle, the sides, and the angles; and in order to solve them, fewer than three of these parts will be insufficient; but, knowing any three, the others may be determined from them by the usual algebraical process of elimination; yet, as in the general formulas for the solution of plane triangles, so here, the results thus obtained would require considerable modification in certain cases to fit them for logarithmic computation, and on this account it is better to deduce particular formulas by a less direct process. Thus, in order to ascertain the relation between the sides and opposite angles of a spherical triangle, we proceed as follows.

(46.) From the equation (A)

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \dots (1)$$

$$\therefore \sin. A = \sqrt{1 - \cos.^2 A} = \frac{\sqrt{\sin.^2 b \sin.^2 c - (\cos. a - \cos. b \cos. c)^2}}{\sin. b \sin. c}$$

or, since

$$\sin.^2 b \sin.^2 c = (1 - \cos.^2 b) (1 - \cos.^2 c)$$

$$\sin. A = \frac{\sqrt{1 - \cos.^2 a - \cos.^2 b - \cos.^2 c + 2 \cos. a \cos. b \cos. c}}{\sin. b \sin. c} \dots (2)$$

$$\therefore \frac{\sin. A}{\sin. a} = \frac{\sqrt{1 - \cos.^2 a - \cos.^2 b - \cos.^2 c + 2 \cos. a \cos. b \cos. c}}{\sin. a \sin. b \sin. c}$$

Now the second side of this equation is plainly of such a form, that, however we interchange the quantities a, b, c , the value of the expres-

sion remains unaltered; so that if we had set out with $\cos. B$, as given by the second of (A), instead of with $\cos. A$, we should have had the very same result for $\frac{\sin. B}{\sin. b}$; hence

$$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c} \dots (3);$$

that is, *in any spherical triangle the sines of the sides are to each other as the sines of the opposite angles*; so that when two of the three given quantities are a side and its opposite angle, the unknown, which is opposite to the third given quantity, may be determined by a simple proportion, or by an easy logarithmic process.

(47.) The equation (2) above might serve to find an angle, from knowing the three sides; it is, however, much less simple than the original expression (1), but neither of them are adapted to logarithms.

In order to obtain one that is adapted, add 1 to each member of (1), and there results (form 24, p. 57),

$$1 + \cos. A = 2 \cos.^2 \frac{1}{2} A = \frac{\cos. a + \sin. b \sin. c - \cos. b \cos. c}{\sin. b \sin. c} \\ = \frac{\cos. a - \cos. (b + c)}{\sin. b \sin. c};$$

but a and $b + c$ are respectively the difference and sum of the two arcs

$$\frac{1}{2} (a + b + c), \text{ and } \frac{1}{2} (b + c - a);$$

hence (form 4, p. 46),

$$\cos. a - \cos. (b + c) = 2 \sin. \frac{1}{2} (a + b + c) \sin. \frac{1}{2} (b + c - a);$$

therefore, putting S for the sum of the three sides, we have

$$\cos. \frac{1}{2} A = \sqrt{\frac{\sin. \frac{1}{4} S \sin. (\frac{1}{2} S - a)}{\sin. b \sin. c}} \dots (1).$$

If instead of adding 1 to each side of (1) p. 76, we subtract each side from 1, and proceed as above, we shall obtain for $\sin. \frac{1}{2} A$ the value,

$$\sin. \frac{1}{2} A = \sqrt{\frac{\sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}{\sin. b \sin. c}} \dots (2);$$

and, by dividing this equation by the former, we have

$$\tan. \frac{1}{2} A = \sqrt{\frac{\sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a)}} \dots (3);$$

and all these expressions are adapted to logarithms.

It is unnecessary to put down the corresponding expressions for the other angles, as they may be obtained from these by simply changing the letters: thus for $\sin. \frac{1}{2} B$, we have, by changing A for B and b for a in (2), the formula

$$\sin. \frac{1}{2} B = \sqrt{\frac{\sin. (\frac{1}{2} S - a) \sin. (\frac{1}{2} S - c)}{\sin. a \sin. c}},$$

whence

$$\frac{\sin. \frac{1}{2} A}{\sin. \frac{1}{2} B} = \sqrt{\frac{\sin. a \sin. (\frac{1}{2} S - b)}{\sin. b \sin. (\frac{1}{2} S - a)}};$$

from which it appears that if $a > b$, $\sin. \frac{1}{2} A > \sin. \frac{1}{2} B$, and therefore $A > B$; also if $b > a$, $\sin. \frac{1}{2} B > \sin. \frac{1}{2} A$; and therefore $B > A$. Consequently *the greater side is always opposite to the greater angle*.

If $b = c$, the equation (2) becomes

$$\sin. \frac{1}{2} A = \frac{\sin. \frac{1}{2} a}{\sin. b}.$$

(48.) We have thus got convenient formulas for the determination of the unknown parts, when two sides and an opposite angle are given, when two angles and an opposite side are given, and when all the three sides are given. We shall now seek the solution to the case in which two sides and the included angle are given, or two angles and the interjacent side; that is to say, we shall proceed to deduce an equation involving only the four quantities a , b , A and C .

For $\cos. c$ in the first of equations (A) substitute its value, as given by the third, and there results, after putting $1 - \sin.^2 b$, for its equal $\cos.^2 b$,

$$\begin{aligned} \cos. a = \cos. a - \cos. a \sin.^2 b + \sin. a \sin. b \cos. b \cos. C + \\ \sin. b \sin. c \cos. A; \end{aligned}$$

or, cancelling $\cos. a$ on each side, dividing by $\sin. b$, and transposing,

$$\cos. a \sin. b = \sin. a \cos. b \cos. C + \sin. c \cos. A \dots (1).$$

For $\sin. c$ in this equation substitute its value given by (3, p. 77), viz.

$$\sin. c = \frac{\sin. a \sin. C}{\sin. A};$$

and it becomes

$$\cos. a \sin. b = \sin. a \cos. b \cos. C + \frac{\sin. a \sin. C \cos. A}{\sin. A};$$

that is, dividing by $\sin. a$,

$$\cot. a \sin. b = \cos. b \cos. C + \sin. C \cot. A;$$

which is the equation we proposed to deduce, and from which we at once get an expression for $\cot. A$, when the two sides a, b , and their included angle C , are given, or for $\cot. a$ when the two angles A, C , and interjacent side b are given. The remaining parts of the triangle may, obviously, be found by the relation (p. 77) between the sides and opposite angles; but if the third side, in terms of the other two, and the included angle, is required in a single formula, we must then recur to the fundamental equations (A), which obviously furnish that formula. But neither this nor that which we have just deduced are calculable by a single logarithmic operation; by the introduction, however, of a subsidiary arc the solution may be conducted by logarithms, although two operations will be necessary. But we shall explain this artifice in the next chapter, which will contain the practical application of the formulas deduced in this.

(49.) It now only remains for us to furnish a formula for the side of a spherical triangle in terms of the three angles, and this we may easily do by help of the formulas already given for an angle in terms of the sides, availing ourselves of the property of the supplemental triangle, viz. that the angles and sides of this are supplements of the sides and angles of the former (41). For let the formulas (47) refer to the supplemental triangle of that in question, then, by marking the letters of the former with an accent for distinction sake, we have

$$A' = 180^\circ - a, a' = 180^\circ - A, b' = 180^\circ - B, c' = 180^\circ - C, S' = 540^\circ - S;$$

S' being the sum of the sides of the triangle in (47), and S the sum of the angles of the triangle with which we are now occupied.

Consequently,

$$\cos. \frac{1}{2} A' = \cos. (90^\circ - \frac{1}{2} a) = \sin. \frac{1}{2} a, \sin. b' = \sin. (180^\circ - B) = \sin. B$$

$$\sin. c' = \sin. (180^\circ - C) = \sin. C, \sin. \frac{1}{2} S' = \sin. (270^\circ - \frac{1}{2} S) =$$

$$- \cos. \frac{1}{2} S$$

$$\sin. (\frac{1}{2} S' - a') = \sin. [90^\circ - (\frac{1}{2} S - A)] = \cos. (\frac{1}{2} S - A);$$

therefore, by substituting these values the formula (2) becomes

$$\sin. \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} S \cos. (\frac{1}{2} S - A)}{\sin. B \sin. C}};$$

and the other two become

$$\cos. \frac{1}{2} a = \sqrt{\frac{\cos. (\frac{1}{2} S - B) \cos. (\frac{1}{2} S - C)}{\sin. B \sin. C}}$$

$$\tan. \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} S \cos. (\frac{1}{2} S - A)}{\cos. (\frac{1}{2} S - B) \cos. (\frac{1}{2} S - C)}}.$$

As $\frac{1}{2}S$ exceeds 90° but falls short of 270° art. (42), $\cos. \frac{1}{2}S$ is always negative, and, therefore, the numerators, of the first and third of these expressions, although appearing with a negative sign, are in reality positive.

(50.) By means of the polar triangle it is obvious that we may, in all cases as well as in this, convert any formula involving the sides and angles of a triangle into another, similarly involving the angles and sides; the sides in the one formula being replaced by the angles opposite to them in the other, and the angles being replaced by the opposite sides. To effect this change we need only write, instead of *sin.* and *cos.* in the original formula, *sin.* and $- \cos.$ of the opposite arc, whether side or angle.

Thus the fundamental equations (A) become in this manner changed into the following,

$$\left. \begin{aligned} \cos. A &= \cos. a \sin. B \sin. C - \cos. B \cos. C \\ \cos. B &= \cos. b \sin. A \sin. C - \cos. A \cos. C \\ \cos. C &= \cos. c \sin. A \sin. B - \cos. A \cos. B \end{aligned} \right\} \dots (B).$$

which plainly show that *if the three angles of one triangle are equal to the three angles of another, the sides of the former must also be equal to those of the latter*; and also that *if two angles B, C, and interjacent side, a, of one triangle are respectively equal to two angles, and the interjacent side of another, the remaining angle A of the one must be equal to the remaining angle of the other*; and thus all the parts of the one triangle are equal severally to those of the other.

(51.) The theory now delivered is sufficient for the solution of every case of spherical triangles; but we shall add two more theorems applicable to the case in which the two sides and included angle are given to find the other angles, and to that in which two angles and the interjacent side are given to find the other sides. These theorems have the advantage of being very simple, and are of a form easily retained in the memory. They were first given by *Lord Napier*, and are known by the name of *Napier's Analogies*.

By the equation (1), page ~~77~~⁷⁹, we have

$$\sin. c \cos. A = \cos. a \sin. b - \sin. a \cos. b \cos. C.$$

Similarly,

$$\sin. c \cos. B = \cos. b \sin. a - \sin. b \cos. a \cos. C$$

$$\therefore \sin. c (\cos. A + \cos. B) = \sin. (a + b) (1 - \cos. C) \dots (1).$$

Now from the equations (3), page 77, we have

$$\sin. A \sin. c = \sin. a \sin. C$$

$$\sin. B \sin. c = \sin. b \sin. C$$

$$\therefore (\sin. A \pm \sin. B) \sin. c = (\sin. a \pm \sin. b) \sin. C \dots (2).$$

Dividing (2) by (1) there results

$$\frac{\sin. A \pm \sin. B}{\cos. A + \cos. B} = \frac{\sin. a \pm \sin. b}{\sin. (a + b)} \cdot \frac{\sin. C}{1 - \cos. C}$$

that is, arts. (32) and (31) taking the upper and lower signs separately,

$$\tan. \frac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} (a + b)} \cot. \frac{1}{2} C$$

$$\tan. \frac{1}{2} (A - B) = \frac{\sin. \frac{1}{2} (a - b)}{\sin. \frac{1}{2} (a + b)} \cot. \frac{1}{2} C.$$

For the supplemental triangle the corresponding formulas are

$$\tan. \frac{1}{2} (a + b) = \frac{\cos. \frac{1}{2} (A - B)}{\cos. \frac{1}{2} (A + B)} \tan. \frac{1}{2} c$$

$$\tan. \frac{1}{2} (a - b) = \frac{\sin. \frac{1}{2} (A - B)}{\sin. \frac{1}{2} (A + B)} \tan. \frac{1}{2} c;$$

and these are the four equations which furnish the Analogies of Napier, viz.

$$\left. \begin{array}{l} \cos. \frac{1}{2} (a + b) : \cos. \frac{1}{2} (a - b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A + B) \\ \sin. \frac{1}{2} (a + b) : \sin. \frac{1}{2} (a - b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A - B) \end{array} \right\} \dots (2)$$

$$\left. \begin{array}{l} \cos. \frac{1}{2} (A + B) : \cos. \frac{1}{2} (A - B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a + b) \\ \sin. \frac{1}{2} (A + B) : \sin. \frac{1}{2} (A - B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a - b) \end{array} \right\} \dots (3).$$

As the arcs $\frac{1}{2} (a - b)$, and $\frac{1}{2} C$, are always less than 90° , the two means in the first of these analogies are positive, and, therefore, the two extremes must have the same signs, that is, they must either be both positive or both negative: hence $\frac{1}{2} (a + b)$, and $\frac{1}{2} (A + B)$, must either be both acute or both obtuse, and consequently the arcs $a + b$, $A + B$, must be either both less or both greater than 180° . From this circumstance we may always avoid doubtful solutions to the cases in which the given parts are two sides and an opposite angle, or two angles and an opposite side, as will be exemplified in next chapter.

CHAPTER III.

SOLUTIONS OF THE DIFFERENT CASES OF SPHERICAL TRIANGLES.

(52.) We are now to show the application of the preceding theory to the actual determination of any of the six parts of a spherical triangle when three of them are known; and as in Plane Trigonometry, so here, we shall find it convenient to begin with right-angled triangles.

Right-Angled Spherical Triangles.

The formulas from which all the rules for right-angled triangles are derived are those marked (A), (B), and 3, (p. 77), in the preceding chapter, viz.

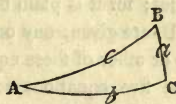
$$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c} \dots (1)$$

$$\left. \begin{aligned} \cos. A &= \cos. a \sin. B \sin. C - \cos. B \cos. C \\ \cos. B &= \cos. b \sin. A \sin. C - \cos. A \cos. C \\ \cos. C &= \cos. c \sin. A \sin. B - \cos. A \cos. B \end{aligned} \right\} \dots (2)$$

$$\left. \begin{aligned} \cos. a &= \cos. b \cos. c + \sin. b \sin. c \cos. A \\ \cos. b &= \cos. a \cos. c + \sin. a \sin. c \cos. B \\ \cos. c &= \cos. a \cos. b + \sin. a \sin. b \cos. C \end{aligned} \right\} \dots (3).$$

Let ABC be a spherical triangle, right-angled at C; then from the first of these formulas we have, since $\sin. C = 1$, the equations

$$\sin. a = \sin. c \sin. A, \sin. b = \sin. c \sin. B. (4).$$



Two different expressions for $\sin. a$, $\sin. b$, may also be obtained

from the first and second of (2). Thus C being 90° these two equations give

$$\cos. A = \cos. a \sin. B, \cos. B = \cos. b \sin. A \dots (5);$$

substituting in these the values of $\sin. A, \sin. B$, as deduced from (1) they become

$$\cos. A = \frac{\cos. a}{\sin. a} \sin. A \sin. b, \cos. B = \frac{\cos. b}{\sin. b} \sin. B \sin. a$$

$$\therefore \sin. b = \tan. a \cot. A, \sin. a = \tan. b \cot. B.$$

For the hypotenuse c we get from the third of (2) the expression

$$\cos. c = \frac{\cos. A \cos. B}{\sin. A \sin. B} = \cot. A \cot. B,$$

and from the third of (3) the expression

$$\cos. c = \cos. a \cos. b \dots (6).$$

In the equations (5) substitute for $\sin. A, \sin. B$, their values in (4), and for $\cos. a, \cos. b$, their values in (6), and they then take the form

$$\cos. A = \tan. b \cot. c, \cos. B = \tan. a \cot. c \dots (7).$$

Collecting together all these equations, we have

$$\sin. a = \tan. b \cot. B = \sin. c \sin. A$$

$$\sin. b = \tan. a \cot. A = \sin. c \sin. B$$

$$\cos. c = \cot. A \cot. B = \cos. a \cos. b$$

$$\cos. A = \tan. b \cot. c = \cos. a \sin. B$$

$$\cos. B = \tan. a \cot. c = \cos. b \sin. A;$$

and these furnish solutions to every possible case of right-angled triangles; for it is plain that whichever two of the *five* quantities a, b, c, A, B , are given, any one of the others may be immediately found by one or other of these equations. Instead, however, of deducing from these five equations so many distinct rules for the solution of the various cases, the whole, by help of an ingenious contrivance, may be comprehended in two rules of very remarkable simplicity.

Before announcing these rules we shall, however, just stop to mention an inference from the first of this group of equations which will be useful hereafter, viz. that *from any point on a sphere to a given great circle the shortest great circle arc that can be drawn is the perpendicular*; for by the equation referred to $\sin. a$ exceeds $\sin. c$, since $\sin. A$ is less than 1. If the point is the pole of the proposed great circle, then, indeed, (p. 67) $\sin. a = \sin. c$, and $\sin. A = 1$, all great circle arcs from the point to the circle being perpendicular.

From the last of the preceding equations we infer that $\cos. B$, $\cos. b$, always have the same sign, that is, either side is of the same affection as its opposite angle. From the middle equation we see that the hypotenuse is acute if the sides are of the same affection, or if the angles opposite to them are of the same affection, but otherwise the hypotenuse is obtuse.

The rules to which we have adverted above were invented by Baron Napier, the celebrated inventor of logarithms, and are called *Napier's Rules for the Circular Parts*. We shall now explain them.

In a right-angled triangle we are to recognize but five parts, viz. the three sides and the two angles A and B . If we take any one of these as a middle part the two which lie next to it, one on each side, will be *adjacent* parts: thus taking A for a middle part (last fig.), b and c will be the *adjacent* parts; if we take c for the middle part, A and B will be the *adjacent* parts; if we take B for the middle part c and a will be the *adjacent* parts; but if we take a for the middle part, then, as the part C is not recognized we do not consider it as intervening between a and b , and, therefore, we call in this case B and b , the *adjacent* parts; and, lastly, if b is the middle part then the *adjacent* parts are A and a . The two parts immediately beyond the *adjacent* parts, one on each side, still disregarding the right-angle, are called the *opposite* parts; thus if A is the middle part the *opposite* parts are a , next to the *adjacent* part b , and B next to the *adjacent* part c . This being understood, Napier's two rules may be expressed as follows, carefully observing to *use the complements of the two angles and of the intervening hypotenuse instead of these parts themselves*.

I.

Rad. $\times$ $\sin.$ middle part = product of $\tan.$ *adjacent* parts.

II.

Rad. $\times$ sin. middle part = product of cos. opposite parts.

Both these rules may be comprehended in a single expression, thus

$$\text{rad. sin. mid.} = \text{prod. tan. adja.} = \text{prod. cos. opp.};$$

and to retain this in the memory we have only to remember that the vowels in the contractions mid., adja., opp., are the same as those in the contractions sin., tan., cos., to which they are joined.

That these rules comprehend all the equations given above will be seen by taking a, b, c , &c. in succession for the middle part, as in the subjoined table, keeping in mind the condition just stated, that instead of A, B , and c , we are to use their complements.

Middle part.	Adjacent parts.	Opposite parts.
(sin.)	(tan.)	(cos.)
a	b , comp. B	comp. c , comp. A
b	a , comp. A	comp. c , comp. B
comp. c	comp. A , comp. B	a, b
comp. A	b , comp. c	a , comp. B
comp. B	a , comp. c	b , comp. A

As in the solution of right-angled triangles two parts are given to find a third, we must in the application of Napier's rule choose for the middle of these three parts that which causes the other two to become either adjacent parts or opposite parts.

EXAMPLES.

(53.) 1. In the right-angled triangle ABC are given the two perpendicular sides, viz. $a = 48^\circ 24' 16''$, $b = 59^\circ 38' 27''$, to find the hypotenuse c .

Here the hypotenuse being made the middle part the other two will, obviously, be the opposite parts, being separated from the hypotenuse by the intervening angles A, C. Hence by the rule

$$\text{rad.} \times \sin. \text{comp. } c = \cos. a \times \cos. b;$$

that is,

$$\text{rad.} \cos. c = \cos. a \cos. b \therefore \cos. c = \frac{\cos. a \cos. b}{\text{rad.}};$$

and as $\cos. a$, $\cos. b$, are both positive, $\cos. c$ is positive, and, therefore, c is acute.

rad.	.	.	.	.	.	10.0000000
$\cos. a$		$48^{\circ} 24' 16''$	.	.	.	9.8220819
$\cos. b$		59 38 27	.	.	.	9.7036515
$\cos. c$		70 23 42	.	.	.	<u>9.5257334.</u>

2. In the spherical triangle ABC, right-angled at C, are given $b = 46^{\circ} 18' 23''$, $A = 34^{\circ} 27' 39''$, to find the other oblique angle B.

Making B the middle part, the other two will be the opposite parts. Consequently, by the rule,

$$\text{rad.} \times \sin. \text{comp. } B = \cos. b \times \cos. \text{comp. } A;$$

that is,

$$\text{rad.} \cos. B = \cos. b \sin. A \therefore \cos. B = \frac{\cos. b \sin. A}{\text{rad.}};$$

and as $\cos. b$, $\sin. A$, are both positive, B is acute,

rad.	.	.	.	.	.	10.0000000
$\cos. b$		$46^{\circ} 18' 23''$	.	.	.	9.8393535
$\sin. A$		34 27 39	.	.	.	9.7526957
$\cos. B$		66 59 25	.	.	.	<u>6.5920492.</u>

3. In the spherical triangle, right-angled at C, are given the two perpendicular sides, viz. $a = 116^{\circ} 30' 43''$, $b = 29^{\circ} 41' 32''$, to find the angle A.

Making b the middle part, the others will be the adjacent parts, and, therefore, by the rule

$$\text{rad.} \times \sin. b = \tan. a \times \tan. \text{comp. } A,$$

that is,

$$\text{rad. sin. } b = \tan. a \cot. A \therefore \cot. A = \frac{\text{rad. sin. } b}{\tan. a};$$

and as $\sin. b$ is positive, and $\tan. a$ negative, $\cot. A$ will be negative, and, therefore, A will be obtuse, or the supplement of the angle given by the tables,

rad.	.	.	.	.	.	10.0000000
$\tan. a$	116° 30' 43"	.	.	.	.	10.3020371
$\sin. b$	29 41 32	.	.	.	.	9.6949041
$\cot. A$	103 52 48	.	.	.	.	<u>9.3928670.</u>

4. In a spherical triangle, right-angled at C , are given $b = 29^\circ 12' 50''$, and $B = 37^\circ 26' 21''$, to find the side a .

Taking a for the middle part, the other two will be adjacent parts; hence, by the rule,

$$\text{rad.} \times \sin. a = \tan. b \times \tan. \text{comp. } B$$

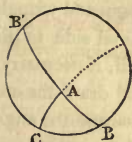
that is,

$$\text{rad. sin. } a = \tan. b \cot. B \therefore \sin. a = \frac{\tan. b \cot. B}{\text{rad.}}$$

In this case there are two solutions, viz. a and the supplement of a , both of which have the same sine. As $\sin. a$ is necessarily positive, b and B must necessarily be always of the same species, that is, either both acute or both obtuse, so that, as observed at p. 85, the sides including the right-angle are always of the same species as the opp. angles, a circumstance which must be attended to in framing examples,

rad.	.	.	.	.	.	10.0000000
$\tan. b$	29° 12' 50"	.	.	.	.	9.7475666
$\cot. B$	37 26 21	.	.	.	.	<u>10.1159745</u>
$\sin. a$	46° 55' 2", or 133° 4' 58"	.	.	.	.	<u>9.8635411.</u>

It appears, therefore, that there exists two right-angled triangles, having an oblique angle, and the opposite side in one equal to an oblique angle and the opposite side in the other, but the remaining oblique angle in the one the supplement of the remaining oblique angle in the other. These triangles are situated, with respect to each other, on the sphere, as the triangles ABC , $AB'C$, in the annexed diagram, in which, with the exception of the common side AC , and the equal angles B , B' , the parts of the one triangle are supplements of the corresponding parts of the other.



5. Given the angle $A = 23^\circ 28'$, the side $b = 49^\circ 17'$, to find the hypotenuse c .

$$c = 51^\circ 42' 37''.$$

6. Given the hypotenuse $c = 66^\circ 32'$, the side $a = 37^\circ 48'$, to find the angle B .

$$B = 70^\circ 19' 18''.$$

7. Given the perpendicular sides $a = 59^\circ 38' 27''$, $b = 48^\circ 24' 16''$, to find all the other parts.

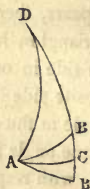
$$c = 70^\circ 23' 42'', A = 66^\circ 20' 40'', B = 52^\circ 32' 55''.$$

8. Given $b = 121^\circ 26' 25''$, and the opposite angle $B = 111^\circ 14' 37''$, to find all the other parts.

Solution of Quadrantal Triangles.

(54.) The rules for right-angled triangles will serve also for the solution of *quadrantal triangles*, or those in which one side is a quadrant. For by changing such a triangle for its supplemental triangle we shall then have to consider a right-angled triangle, of which the hypotenuse will be the supplement of the angle opposite the quadrantal side, the two perpendicular sides supplements of the other two angles of the proposed triangle, and the two oblique angles of the new triangle supplements of the oblique sides of the primitive triangle. That is, the sides of the primitive or quadrantal triangle being a , b , and $c = 90^\circ$, and its angles A , B , C , the sides of the supplemental triangle will be $180^\circ - A$, $180^\circ - B$, and $180^\circ - C$, this latter being the hypotenuse; and the opposite angles will be $180^\circ - a$, $180^\circ - b$, and 90° . But the parts of a quadrantal triangle may be determined without the aid of the sup-

plemental triangle. Thus let AD be the quadrantal side in the triangle ABD . Produce DB , if necessary, till DC becomes a quadrant, and draw the arc AC , which will, obviously, measure the angle D , since D will be the pole of the arc AC , and C will be a right-angle: also the angle CAB will be the complement of the angle BAD in the proposed triangle, and the angle ABC will either be identical with ABD in the proposed, or supplemental to it, accordingly as DC exceeds, or falls short of, a quadrant; hence all the parts of the proposed triangle are easily determined from those of the right-angled triangle ABC .



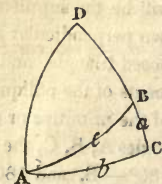
If the angle DAB is less than 90° , or than the angle DAC , the side DB must, obviously, be acute; but if DAB is greater than 90° , DB will be obtuse, and conversely. Hence the angles adjacent to the quadrantal side are of the same species as the sides opposite to them. The same may be inferred from the polar triangle.

It must be remarked that the solution will be ambiguous whenever the determination of the right-angled triangle becomes ambiguous, whether we employ the polar triangle or the triangle ABC in the above diagram. This ambiguity occurs only when the given parts in the right-angled triangle are one of the perpendicular sides and the angle opposite to it. (See ex. 4, p. 88.)

EXAMPLES.

In the triangle DAB , $DA = 90^\circ$, $A = 54^\circ 43'$, and $D = 42^\circ 12'$, required the other parts.

As the angle DAB is less than 90° , that is, less than the angle DAC , DB is less than a quadrant, and, therefore, the right-angled triangle ABC is situated as in the figure, BC being the prolongation of DB . Of the parts of this right-angled triangle we have given $A = 90^\circ - 54^\circ 43' = 35^\circ 17'$, and $b = 42^\circ 12'$, to find the other parts.



Let A be the middle part, then b and c will be adjacent parts,

therefore,

$$\text{rad.} \times \sin. \text{comp. } A = \tan. b \times \tan. \text{comp. } c,$$

that is,

$$\text{rad. cos. } A = \tan. b \cot. c \therefore \cot. c = \frac{\text{rad. cos. } A}{\tan. b}$$

rad.	.	.	.	.	.	10.0000000
cot. A	35° 17'	.	.	.	.	9.9118528
tan. b	42 12	.	.	.	.	9.9574850
cot. c	48° 0' 9"	.	.	.	.	<u>9.9543678.</u>

Let B be the middle part, then A, b, will be opposite parts, and, consequently,

$$\text{rad.} \times \sin. \text{comp. } B = \cos. b \times \cos. \text{comp. } A;$$

that is,

$$\text{rad. cos. } B = \cos. b \sin. A \therefore \cos. B = \frac{\cos. b \sin. A}{\text{rad.}}$$

rad.	.	.	.	.	.	10.0000000
cos. b	42° 12'	.	.	.	.	9.8697037
sin. A	35 17	.	.	.	.	9.7616424
cos. B	64° 39' 55"	.	.	.	.	<u>9.6313461;</u>

hence the angle ABD is 115° 20' 5".

It remains now to find a; let, therefore, B be the middle part, then a and c will be the adjacent parts; hence

$$\text{rad.} \times \sin. \text{comp. } B = \tan. a \times \tan. \text{comp. } c;$$

that is,

$$\text{rad. cos. } B = \tan. a \cot. c \therefore \tan. a = \frac{\text{rad. cos. } B}{\cot. c}$$

rad.	.	.	.	.	.	10.0000000
cos. B	.	.	.	.	.	9.6313461
cot. c	.	.	.	.	.	9.9543678
tan. a	25° 25' 20"	.	.	.	.	<u>9.6769783;</u>

therefore, the side DB, which is the complement of this, is $64^{\circ} 34' 40''$.

2. In the triangle DAB, $DA = 90^{\circ}$, $A = 112^{\circ} 2' 9''$, and $AB = 67^{\circ} 3' 14''$, to find the other parts.

Since in this example A is obtuse, DB is obtuse.

In the right-angled triangle ABC we have $A = 22^{\circ} 2' 9''$ and $AB = 67^{\circ} 3' 14''$; let A be the middle part, then AB, AC, will be adjacent parts, and we shall have

$$\text{rad.} \times \sin. \text{comp. } A = \tan. b \times \tan. \text{comp. } c;$$

that is,

$$\text{rad.} \cos. A = \tan. b \cot. c \therefore \tan. b = \frac{\text{rad.} \cos. A}{\cot. c}$$

rad.	.	.	.	.	.	10.0000000
cos. A	$22^{\circ} 2' 9''$	.	.	.	.	9.9670560
cot. c	$67 \ 3 \ 14$	.	.	.	.	9.6267152
tan. b	$65 \ 27 \ 9$	.	.	.	.	10.3403408;

therefore, the angle $D = 65^{\circ} 27' 9''$.

Take now a for the middle part, then A and c will be opposite parts; hence

$$\text{rad.} \times \sin. a = \cos. \text{comp. } A \times \cos. \text{comp. } c,$$

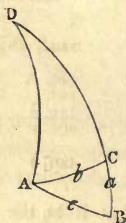
that is,

$$\text{rad.} \sin. a = \sin. A \sin. c \therefore \sin. a = \frac{\sin. A \sin. c}{\text{rad.}};$$

and a will be acute, because the opposite angle is acute

rad.	.	.	.	.	.	10.0000000
sin. A	$22^{\circ} 2' 9''$	.	.	.	.	9.5742471
sin. c	$67 \ 3 \ 14$	.	.	.	.	9.9641992
sin. a	$22 \ 12 \ 44$	.	.	.	.	9.5384463;

therefore $BD = 110^{\circ} 12' 44''$.



As we have now to find B , take a for the middle part, then b and B will be adjacent parts, therefore

$$\text{rad.} \times \sin. a = \tan. b \tan. \text{comp. } B;$$

that is,

$$\text{rad.} \sin. a = \tan. b \cot. B \therefore \cot. B = \frac{\text{rad.} \sin. a}{\tan. b}$$

rad.	.	.	.	.	.	10.0000000
sin. a	.	.	.	.	.	9.5384463
tan. b	.	.	.	.	.	10.3403408
cot. B	81° 1' 58"	.	.	.	.	9.1981055

3. Given the quadrantal side and the other two sides equal to $22^\circ 53' 30''$, and $51^\circ 4' 35''$, to find the angle opposite to the quadrantal side.

$$B = 70^\circ 3' 44''.$$

4. In the quadrantal triangle ADB are given $D = 69^\circ 13' 46''$, and $A = 72^\circ 12' 4''$, to determine the other parts.

$$AB = 70^\circ 8' 39'', \quad BD = 73^\circ 17' 29'', \quad B = 96^\circ 13' 23''.$$

These examples will suffice, for the present, to show the application of Napier's rules to the solution of right-angled and quadrantal triangles. We shall, therefore, now give examples of the solution of the various cases of oblique angled triangles in general.

Solution of Oblique Angled Spherical Triangles.

(55.) The fundamental equations (A) show that in order to determine the several parts of a spherical triangle, three of those parts must be previously given. Now, three parts out of the six can be combined only in these different ways, viz.

1. The three sides.
2. The three angles.
3. Two sides and the included angle.
4. Two angles and the interjacent side.
5. Two sides and an opposite angle.
6. Two angles and an opposite side.

So that the complete solution of an oblique angled spherical triangle presents six cases. These we shall solve in the order in which they are here enumerated.

CASE I.

(56.) Given the three sides to find the angles.

For the determination of any angle A we have by (47) the three following different expressions, viz.

$$\sin. \frac{1}{2} A = \sqrt{\frac{\sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}{\sin. b \sin. c}}$$

$$\cos. \frac{1}{2} A = \sqrt{\frac{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a)}{\sin. b \sin. c}}$$

$$\tan. \frac{1}{2} A = \sqrt{\frac{\sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a)}}$$

We may apply to these formulas the remarks made at (21) in the Plane Trigonometry. It will be sufficient to observe here that the first formula is generally the most suitable, because the angle A is rarely so large as to be very near 180° .

EXAMPLES.

1. In an oblique spherical triangle the three sides are

$$a = 68^\circ 46' 2'', \quad b = 43^\circ 37' 38'', \quad c = 37^\circ 10';$$

required the angle A .

$$a \quad 68^{\circ} 46' 2''$$

$$\sin. b \quad 43 \quad 37 \quad 38 \text{ arith. comp. } 0.1611739$$

$$\sin. c \quad 37 \quad 10 \quad 0 \text{ arith. comp. } 0.2188656$$

$$2) 149 \quad 33 \quad 40$$

$$74 \quad 46 \quad 50$$

$$\sin. (\tfrac{1}{2} S - b) \quad 31 \quad 9 \quad 12 \quad . \quad . \quad 9.7137678$$

$$\sin. (\tfrac{1}{2} S - c) \quad 37 \quad 36 \quad 50 \quad . \quad . \quad 9.7855698$$

$$2) 19.8793771$$

$$\sin. \tfrac{1}{2} A \quad 60 \quad 29 \quad 53 \quad . \quad . \quad 9.9396885$$

$$\therefore A = 120^{\circ} 59' 46''.$$

2. Given $a = 108^{\circ}$, $b = 52^{\circ} 12'$, and $c = 74^{\circ} 30'$, to find A .

$$a \quad 108^{\circ} 0'$$

$$\sin. b \quad 37 \quad 48 \text{ arith. comp. } 0.2126054$$

$$\sin. c \quad 74 \quad 30 \text{ arith. comp. } 0.0160895$$

$$2) 220 \quad 18$$

$$110 \quad 9$$

$$\sin. (\tfrac{1}{2} S - b) \quad 72 \quad 21 \quad . \quad . \quad 9.9790594$$

$$\sin. (\tfrac{1}{2} S - c) \quad 35 \quad 39 \quad . \quad . \quad 9.7655436$$

$$2) 19.9732979$$

$$\sin. \tfrac{1}{2} A \quad 75 \quad 51 \quad 56 \quad . \quad . \quad 9.9866489$$

$$\therefore \sin. A = 151^{\circ} 43' 52''.$$

3. Given $a = 70^{\circ} 4' 18''$, $b = 63^{\circ} 21' 27''$, and $c = 59^{\circ} 16' 23''$, to find the angles A and B .

$$A = 81^{\circ} 38' 20'', \quad B = 70^{\circ} 9' 38''.$$

4. Given $a = 67^\circ 25' 2''$, $b = 80^\circ 2' 25''$, $c = 23^\circ 27' 46''$, to find the angle A.

$$A = 54^\circ 55' 19''.$$

5. Given $a = 61^\circ 32' 12''$, $b = 83^\circ 19' 42''$, $c = 23^\circ 27' 46''$, to find A.

$$A = 20^\circ 39' 48''.$$

CASE II.

(57.) Given the three angles to find the sides.

By (49) we have the following formulas for any side a in terms of the three angles, viz.

$$\sin. \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} S \cos. (\frac{1}{2} S - A)}{\sin. B \sin. C}}$$

$$\cos. \frac{1}{2} a = \sqrt{\frac{\cos. (\frac{1}{2} S - B) \cos. (\frac{1}{2} S - C)}{\sin. B \sin. C}}$$

$$\tan. \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} S \cos. (\frac{1}{2} S - A)}{\cos. (\frac{1}{2} S - B) \cos. (\frac{1}{2} S - C)}}$$

It may be remarked here that the first two only of the expressions in this and in the former case need be borne in the memory, as the third is an immediate consequence of them. If the expressions in the former case be recollected, these can scarcely fail to be recalled at the same time, as they differ from them only in this, viz. that the sides are replaced by their opposite angles, and, except in the denominators, cosines are written for sines, and sines for cosines.

EXAMPLES.

1. The three angles of a spherical triangle are,

$$A = 130^\circ 3' 11'', B = 31^\circ 34' 26'', C = 30^\circ 28' 12'',$$

required the side a .

A	130°	3'	11"		
sin. B	31	34	26	arith. comp.	0·2810023
sin. C	30	28	12	arith. comp.	0·2949174

$$\begin{array}{r} 2)192 \quad 5 \quad 49 \\ \hline \end{array}$$

cos. $\frac{1}{2}$ S	96	2	54 $\frac{1}{2}$	.	.	9·0227162
cos. ($\frac{1}{2}$ S - A)	34	0	16 $\frac{1}{2}$	.	.	9·9185570

$$\begin{array}{r} 2)19·5171929 \\ \hline \end{array}$$

sin. $\frac{1}{2}$ a	35	1	1 $\frac{1}{2}$	.	.	9·7585964
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$$\therefore a = 70^{\circ} 2' 3''.$$

2. The three angles of a spherical triangle are,

$$A = 103^{\circ} 59' 57'', \quad B = 46^{\circ} 18' 7'', \quad C = 36^{\circ} 7' 52'';$$

required the side a .

$$a = 42^{\circ} 8' 48''.$$

3. The three angles of a spherical triangle are $120^{\circ} 43' 37''$, $109^{\circ} 55' 42''$, and $116^{\circ} 38' 33''$; required the three sides.

$$115^{\circ} 13' 26'', \quad 98^{\circ} 21' 40'', \quad \text{and} \quad 109^{\circ} 50' 22''.$$

CASE III.

(58.) Given two sides a, b , and the included angle C , to find the other parts.

By Napier's analogies,

$$\cos. \frac{1}{2} (a + b) : \cos. \frac{1}{2} (a \sim b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A + B)$$

$$\sin. \frac{1}{2} (a + b) : \sin. \frac{1}{2} (a \sim b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A \sim B).$$

These serve to determine the angles A, B , opposite to the given sides; after which the third side c may be determined by either of the remaining two analogies of Napier, viz.

$$\cos. \frac{1}{2} (A + B) : \cos. \frac{1}{2} (A \sim B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a + b)$$

$$\sin. \frac{1}{2} (A + B) : \sin. \frac{1}{2} (A \sim B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a - b).$$

EXAMPLES.

1. In a spherical triangle are given $a = 38^\circ 30'$, $b = 70^\circ$, and $C = 31^\circ 34' 26''$, to find the other parts.

I. To find A and B.

$\cos. \frac{1}{2}(a+b)$	$54^\circ 15'$	ar. comp.	0.2334015	ar. comp. sin.	0.0906719
$\cos. \frac{1}{2}(a \sim b)$	15 45	.	9.9833805	sin.	9.4336746
$\cot. \frac{1}{2}C$	15 47 13	.	10.5486352	.	10.5486352
$\tan. \frac{1}{2}(A+B)$	80 15 41	.	10.7654172	$\tan. \frac{1}{2}(A \sim B)$	49° 47' 30" 10.0729817

$\frac{1}{2}(A+B)$ must be acute, because $\frac{1}{2}(a+b)$ is acute.

By taking the sum and difference of these results we have, $B = 130^\circ 3' 11''$, and $A = 30^\circ 28' 11''$.

II. To find c.

$\cos. \frac{1}{2}(A \sim B)$	49° 47' 30"	arith. comp.	0.1900575
$\cos. \frac{1}{2}(A+B)$	80 15 41	.	9.2282812
$\tan. \frac{1}{2}(a+b)$	54 15 0*	.	10.1427296
$\tan. \frac{1}{2}c$	20 0 0	.	9.5610683

$\therefore c = 49^\circ 0' 0''$.

* There will be no necessity to refer to the tables for the tangent of this arc, we shall obtain it by subtracting the right-hand arithmetical complement in the preceding logarithmic process from that on the left, adding 10 to the index. For calling the right-hand complement p , and the left q , and recollecting that

$$\log. \tan. = 10 + \log. \sin. - \log. \cos. = 10 + (10 - p) - (10 - q),$$

we have

$$\log. \tan. = 10 + q - p.$$

When, in the case we are considering, the only part required happens to be the side opposite the given angle, the finding of the other two angles then becomes merely a subsidiary operation, and the determination of the required side, by Napier's analogies, seems somewhat lengthy. But a shorter method of solution is deducible from the fundamental formula,

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C \dots (1).$$

For substituting $\cos. a \tan. a$ for its equal $\sin. a$ it becomes

$$\cos. c = \cos. a (\cos. b + \tan. a \sin. b \cos. C).$$

Assume

$$\tan. a \cos. C = \cot. \omega = \frac{\cos. \omega}{\sin. \omega};$$

then

$$\begin{aligned} \cos. c &= \cos. a \frac{\sin. \omega \cos. b + \sin. b \cos. \omega}{\sin. \omega} \\ &= \frac{\cos. a \sin. (\omega + b)}{\sin. \omega}. \end{aligned}$$

Hence, to find the side c , we must first determine a subsidiary angle ω from the equation

$$\cot. \omega = \tan. a \cos. C \dots (2);$$

after which c is found by the equation

$$\cos. c = \frac{\cos. a \sin. (\omega + b)}{\sin. \omega} \dots (3).$$

2. The same parts being, given as in the last example, to determine c by these formulas,

$\tan. a \ 38^{\circ} 30' \ 0''$	$\dots$	9.9006052	$\cos. a$	$\dots$	9.8935444
$\cos. C \ 31 \ 34 \ 26$	$\dots$	9.9304221	$\sin. \omega$	ar. comp.	0.0820662
$\cot. \omega \ 55 \ 52 \ 30\frac{1}{2}$	$\dots$	9.8310273	$\sin. (\omega + b)$		9.9086437
			$\cos. c \ 40^{\circ}$	$\dots$	9.8842533

Other formulas for the determination of c might be easily deduced from the same equation (1), but this is as short and as convenient as any. We might also introduce here a distinct formula for the determination of one of the angles A , by help of a subsidiary arc ω ; but as little or nothing would be gained, in point of brevity, over the process by Napier's analogies, we shall not stop to investigate it.

3. In a spherical triangle are given

$$a = 43^\circ 37' 38'', \quad b = 37^\circ 10', \quad C = 120^\circ 59' 46'',$$

to find the side c

$$c = 68^\circ 46' 2''.$$

4. In a spherical triangle are given the two sides, equal to $37^\circ 10'$ and $68^\circ 46' 2''$, and the included angle equal to $39^\circ 23'$; required the other two angles.

$$33^\circ 45' 3'' \text{ and } 120^\circ 59' 49''.$$

5. Given the two sides equal to $44^\circ 13' 45''$ and $84^\circ 14' 29''$, and the included angle equal to $36^\circ 45' 28''$; to find the other parts.

The angles are $32^\circ 26' 6''$, and $130^\circ 5' 22''$, and the side $51^\circ 6' 12''$.

CASE IV.

(59.) Given two angles A , B , and the interjacent side c , to find the other parts.

The solution of this case, as well as the former, is comprehended in Napier's analogies; the one pair, viz.

$$\cos. \frac{1}{2} (A + B) : \cos. \frac{1}{2} (A \sim B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a + b)$$

$$\sin. \frac{1}{2} (A + B) : \sin. \frac{1}{2} (A \sim B) :: \tan. \frac{1}{2} c : \tan. \frac{1}{2} (a - b);$$

determining the unknown sides a , b , and either of the other pair, viz.

$$\cos. \frac{1}{2} (a + b) : \cos. \frac{1}{2} (a \sim b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A + B)$$

$$\sin. \frac{1}{2} (a + b) : \sin. \frac{1}{2} (a \sim b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A - B);$$

enabling us to find the unknown angle C .

EXAMPLES.

1. In a spherical triangle are given two angles equal to $39^\circ 23'$, and $33^\circ 45' 3''$, and the interjacent side equal to $68^\circ 46' 2''$; to find the remaining parts.

I. To find the Sides.

$\cos. \frac{1}{2}(A+B)$	$36^\circ 34' 1\frac{1}{2}''$	ar. comp.	0.0951980	ar. comp. sin.	0.2249260
$\cos. \frac{1}{2}(A-B)$	2 48 58 $\frac{1}{2}$	.	9.9994752	sin.	8.6913737
$\tan. \frac{1}{2}c$	34 23 1	.	9.8352429	.	9.8352429
$\tan. \frac{1}{2}(a+b)$	40 23 49	.	9.9299161	$\tan. \frac{1}{2}(a-b)$	$3^\circ 13' 48''$ 8.7515426

$$\therefore a = 43^\circ 37' 37'', b = 37^\circ 10' 1''.$$

II. To find the Angle.

$\sin. \frac{1}{2}(a-b)$	$3^\circ 13' 48''$	arith. comp.	1.2491502
$\sin. \frac{1}{2}(a+b)$	40 23 49	.	9.8116281
$\tan. \frac{1}{2}(A-B)$	2 48 58 $\frac{1}{2}^*$	.	8.6918985
$\cot. \frac{1}{2}C$	60 29 53	.	9.7526768

$$\therefore C = 120^\circ 59' 46''.$$

If the angle opposite to the given side be the only part required, a more compendious method of solution may be obtained by introducing a subsidiary arc, as in last case. Thus the formula (B) art. (50) becomes when $\cos. A \tan. A$ is substituted for $\sin. A$,

$$\cos. C = \cos. A (\tan. A \sin. B \cos. c - \cos. B);$$

* The log. tangent of this arc will be equal to log. sin. — log. cos., before given, increased by 10.

or assuming

$$\tan. A \cos. c = \cot. \omega = \frac{\cos. \omega}{\sin. \omega}$$

$$\begin{aligned} \cos. C &= \cos. A \frac{\sin. B \cos. \omega - \sin. \omega \cos. B}{\sin. \omega} \\ &= \frac{\cos. A \sin. (B - \omega)}{\sin. \omega}. \end{aligned}$$

Hence, having found a subsidiary angle ω , by the equation

$$\cot. \omega = \tan. A \cos. c \dots (1);$$

the sought angle is determined by the equation

$$\cos. C = \frac{\cos. A \sin. (B - \omega)}{\sin. \omega}.$$

2. The given quantities being the same as in last example, to determine the angle C .

$\tan. A$	$39^{\circ} 23' 0''$	9.9143020	$\cos. A$	.	.	.	9.8881335
$\cos. c$	$68 \ 46 \ 2$	9.5588979	$\sin. \omega$, ar. comp.	.	.	.	0.0183921
$\cot. \omega$	$73 \ 26 \ 33\frac{1}{2}$	9.4731999	$\sin. (B - \omega)$	$39^{\circ} 41' 30\frac{1}{2}''$	.	.	9.8052682
$\cos. 59^{\circ} 0' 13''$							$97117938.$

As $(B - \omega)$ is negative, $\cos. C$ must be negative; hence C is the supplement of this, viz. $120^{\circ} 59' 47''$.

3. Given $A = 30^{\circ} 28' 11''$, $B = 130^{\circ} 3' 11''$, and $c = 40^{\circ}$; to determine the other parts.

$$a = 38^{\circ} 30', b = 70^{\circ}, \text{ and } C = 31^{\circ} 34' 26''.$$

4. Given $A = 31^{\circ} 34' 26''$, $B = 30^{\circ} 28' 12''$, and $C = 70^{\circ} 2' 3''$; to find the angle C .

$$C = 130^{\circ} 3' 11''.$$

5. Given $A = 34^{\circ} 15' 3''$, $B = 42^{\circ} 15' 13''$, and $C = 76^{\circ} 35' 36''$; to find a and b .

$$a = 40^{\circ} 0' 10'', b = 50^{\circ} 10' 30''.$$

6. Given $A = 51^\circ 30'$, $B = 131^\circ 30'$, and $c = 80^\circ 19' 12''$; to find C .

$$C = 59^\circ 15' 59''.$$

CASE V.

(60.) Given two sides a , b , and the angle A opposite to a ; to find the other parts B , C , c .

1. To find the angle B we have, by (46) the proportion,

$$\sin. a : \sin. b :: \sin. A : \sin. B \dots (1).$$

2. To find C and c , we have, by Napier's analogies,

$$\left. \begin{aligned} \cos. \frac{1}{2} (a \sim b) : \cos. \frac{1}{2} (a + b) :: \tan. \frac{1}{2} (A + B) : \cot. \frac{1}{2} C \\ \cos. \frac{1}{2} (A \sim B) : \cos. \frac{1}{2} (A + B) :: \tan. \frac{1}{2} (a + b) : \tan. \frac{1}{2} c \end{aligned} \right\} \dots (2).$$

Or after either C or c is found by one of these analogies the other part may be found by the proportion

$$\sin. A : \sin. C :: \sin. a : \sin. c \dots (3);$$

although we shall prefer Napier's analogy to this in order that all ambiguity may be avoided.

If only one of the parts C , c , be required, then it will be best to find first the angle B , by the proportion (1), which operation must be regarded entirely as subsidiary to the determination of the required part, by one of the analogies (2). The part determined by the proportion (1) admits of a double value, since two arcs answer to the same sine, it becomes necessary, therefore, for us to inquire under what circumstances both these values are admissible, and how we may know which to choose when but one solution exists.

Referring to the fundamental formula (A), we have

$$\cos. B = \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c};$$

in which expression we may remark that if $\cos. b$ is numerically greater than either $\cos. a$ or $\cos. c$, the second member must take the sign of $\cos. b$, consequently, B and b must be of the same species if $\sin. b < \sin.$

a , or $\sin. b < \sin. c$, that is, *an angle must be of the same species as its opposite side, if the sine of this side is less than the sine of either of the other sides.*

But if $\cos. b$ is numerically less than $\cos. a$, then whether the right-hand member be $+$ or $-$ will depend upon the magnitude of $\cos. c$, or $\cos. c$ will have two values corresponding to $+$ $\cos. B$, and $-$ $\cos. B$; hence *an angle has two values, when the sine of its opposite side is greater than the sine of the other given side.*

EXAMPLES.

1. Given the side $a = 63^\circ 50'$, the side $b = 80^\circ 19'$, and the angle $A = 51^\circ 30'$; to determine the other parts.

I. To find the Angle B.

$\sin. a$	$63^\circ 50'$	arith. comp.	0.0469582
: $\sin. b$	$80 19$	.	9.9937679
: : $\sin. A$	$51 30$	.	9.8935444
: $\sin. B$	$59^\circ 15' 47''$	.	9.9342705.

The angle B admits of two values, because $\sin. b > \sin. a$, so that there exist two triangles, having the data proposed.

We shall, however, take the acute value of B.

II. To find the Angle C.

$\cos. \frac{1}{2}(a \sim b)$	$8^\circ 14' 30''$	arith. comp.	0.0045086
: $\cos. \frac{1}{2}(a + b)$	$72 4 30$	.	9.4882288
: : $\tan. \frac{1}{2}(A + B)$	$55 22 53$	.	10.1609412
: $\cot. \frac{1}{2} C$	$65 44 53$	.	9.6536786
$\therefore C = 131^\circ 29' 46''$.			

III. *To find the Side c.*

$\cos. \frac{1}{2}(A \sim B)$	$3^{\circ} 52' 53''$	arith. comp.	0.0009973
: $\cos. \frac{1}{2}(A + B)$	$55 \ 22 \ 53$	.	9.7544333
: : $\tan. \frac{1}{2}(a + b)$	$72 \ 4 \ 30$	.	10.4901618
: $\tan. \frac{1}{2}c$	$60 \ 24 \ 0$	.	10.2455924
	2		
$\therefore c =$	<u>120 48 0.</u>		

2. Given $a = 40^{\circ} 36' 37''$, $b = 91^{\circ} 3' 25''$, and $A = 35^{\circ} 57' 15''$; to determine C.

1. *To find the subsidiary Angle B.*

$\sin. a$	$40^{\circ} 36' 37''$	arith. comp.	0.1864788
: $\sin. b$	$91 \ 3 \ 25$	.	9.9999261
: : $\sin. A$	$35 \ 57 \ 15$	.	9.7687401
: $\sin. B$	$64 \ 24 \ 19$	.	9.9551450
	or 115 35 41		

The angle B admits of two values, because $\sin. b > \sin. a$. We shall suppose the particular triangle under consideration to have B obtuse.

II. *To find C.*

$\cos. \frac{1}{2}(a \sim b)$	$25^{\circ} 13' 24''$	arith. comp.	0.0435177
: $\cos. \frac{1}{2}(a + b)$	$65 \ 50 \ 1$	.	9.6121350
: : $\tan. \frac{1}{2}(A + B)$	$75 \ 46 \ 28$	.	10.5959988
: $\cot. \frac{1}{2}C$	$29 \ 15 \ 28$	.	10.2516515
	2		
$\therefore C =$	<u>58 30 56.</u>		

3. Given $a = 40^\circ 18' 29''$, $b = 67^\circ 14' 28''$, and $A = 34^\circ 22' 17''$; to determine the other parts when B is acute.

$$B = 53^\circ 35' 15'', C = 119^\circ 13' 31'', c = 89^\circ 47' 6''.$$

4. Given $a = 84^\circ 14' 29''$, $b = 44^\circ 13' 45''$, and $A = 130^\circ 5' 22''$; to determine the other parts.

$$B = 32^\circ 26' 6\frac{2}{3}'', C = 36^\circ 45' 28'', c = 51^\circ 6' 12''.$$

5. Given $a = 97^\circ 18' 39''$, $b = 86^\circ 53' 46''$, and $A = 97^\circ 21' 26''$; to determine c .

$$c = 89^\circ 21' 37''.$$

CASE VI.

(61.) Given two angles A, B , and the side a opposite to one of them, to find the other parts.

1. To find b we have

$$\sin. A : \sin. B :: \sin. a : \sin. b.$$

2. And to find C and c we may employ Napier's analogies, which need not be here repeated.

The nature of the arc b may be discussed, as in the preceding case. Thus the formula (B), art. (50), gives

$$\cos. b = \frac{\cos. B + \cos. A \cos. C}{\sin. A \sin. C};$$

from which it follows, as in the foregoing case, that if $\cos. B$ is numerically greater than $\cos. A$, B and b will be of the same species. If $\cos. B$ is numerically less than $\cos. A$, then both the values of b , given by the above proportion, will be admissible, for C may be determined so as to render $\cos. b$ positive or negative. Hence *any side will be of the same species as its opposite angle, if the sine of this angle be less than the sine of either of the other angles; and the species of the side b will be indeterminate if the sine of its opposite angle B be greater than the sine of the other given angle A .* There cannot, therefore, be two solutions unless a and A are of the same species.

EXAMPLES.

1. In an oblique-angled spherical triangle ABC are given, $A = 32^\circ 26' 6\frac{2}{3}''$, $B = 130^\circ 5' 22''$, and the side $a = 44^\circ 13' 42''$; to determine the other parts.

I. To find the Side b .

As sin. A	$32^\circ 26' 6\frac{2}{3}''$	arith. comp.	9.8836842
: sin. B	$130^\circ 5' 22''$	.	0.2705556
:: sin. a	$44^\circ 13' 45''$	.	9.8435629
: sin. b	$84^\circ 14' 29''$	.	<u>9.9978027</u>

b has two values, because the sine of B is greater than that of A. We shall take the acute value.

II. To find the Side c .

As cos. $\frac{1}{2}(A \sim B)$	$48^\circ 49' 37\frac{2}{3}''$	arith. comp.	0.1815543
: cos. $\frac{1}{2}(A + B)$	$81^\circ 15' 44\frac{2}{3}''$	.	9.1815890
:: tan. $\frac{1}{2}(a + b)$	$64^\circ 14' 7''$	.	<u>10.3163591</u>
: tan. $\frac{1}{2}c$	$25^\circ 33' 6''$	.	<u>9.6795024</u>

2

$$\therefore c = 51^\circ 6' 12''$$

III. To find the Angle C.

As cos. $\frac{1}{2}(a \sim b)$	20° 0' 22"	arith. comp.	0.0270310
: cos. $\frac{1}{2}(a + b)$	64 14 7	.	9.6381663
:: tan. $\frac{1}{2}(A + B)$	81 15 44 $\frac{2}{3}$	.	10.8133435
: cot. $\frac{1}{2}C$	18 22 44	.	10.4785408
	2		
$\therefore C =$	36 45 28.		

2. Given $A = 103^\circ 59' 57\frac{1}{2}''$, $B = 46^\circ 18' 7\frac{1}{4}''$, and $a = 42^\circ 8' 48''$; to find the angle C.

$$C = 36^\circ 7' 52\frac{1}{2}''.$$

3. Given $A = 17^\circ 46' 16\frac{1}{2}''$, $B = 151^\circ 43' 52''$, and $a = 37^\circ 48'$; to find the remaining sides, b being obtuse,

$$b = 108^\circ, c = 74^\circ 30'.$$

SCHOLIUM.

Previously to closing this second part it may be worth while to remark, that if, in the foregoing investigations, we consider the radius of the sphere, upon which the triangles concerned are described, to be infinite, then, as any finite portion of the spheric surface may be considered as a plane, the spherical triangles will become plane triangles, and the sines and tangents of their sides will become identical with the sides themselves; so that all the foregoing rules and formulas, into which cosines, cotangents, secants, or cosecants, of the sides do not enter, are applicable as well to plane as to spherical triangles.

* * Before proceeding to the following part, the student may consult note (B), at the end, wherein is examined the unsound doctrine laid down by *Professor Vince*, at page 43 of his *Trigonometry*, (third edition,) respecting the tangent and secant of an arc of 90° .

PART III.

APPLICATION OF PLANE AND SPHERICAL TRIGONOMETRY TO THE PRINCIPLES OF NAVIGATION AND NAUTICAL ASTRONOMY.

(62.) HAVING in the two preceding parts of the present treatise pretty fully explained and illustrated the principles of plane and spherical trigonometry, we shall now, for the purpose of showing the practical utility of these principles, apply them to the solution of one of the most important mathematical problems that has ever engaged the attention of man, viz. *to determine the place of a ship at sea.*

When a ship sails from any known place, and a correct account is kept of her various directions, and rates of sailing, her situation at any time may be readily ascertained by the rules of plane trigonometry, and the solution of the problem from these data belongs to *Navigation*.

But it is impossible to measure a ship's course and the distance sailed exactly; so that after a long passage it would be unsafe to compute the place of the ship from the *ship's reckoning*. In such cases, therefore, the solution must be effected from other data, independent of the ship's account; these are furnished by astronomical observation, and the computation is performed by the rules of spherical trigonometry; the problem then becomes one of *Nautical Astronomy*. We shall devote a distinct chapter to each of these important branches.

CHAPTER I.

THE PRINCIPLES OF NAVIGATION.

Definitions.

(63.) 1. The earth is very nearly spherical. For the purposes of Navigation it may be considered as perfectly so. It revolves round one of its diameters, called its *axis*, in about twenty-four hours. This rotation is from the west towards the east, causing the heavenly bodies to have an apparent motion from the east towards the west.

2. The great circle, whose poles are the extremities of the axis, is called the *equator*. The poles of the equator are called also the poles of the earth; the one being the north pole, and the other the south pole.

3. Every great circle which passes through the poles, and which, therefore, cuts the equator at right-angles, is called a *meridian* circle. Through every place on the surface of the earth such a great circle is supposed to be drawn; it is the meridian of the place. It is expedient for the purposes of Geography and Navigation to fix upon one of these meridians as a *first meridian*, from which the meridians of other places are measured.

The English have fixed upon the meridian of Greenwich Observatory for the first meridian.

4. The longitude of any place is the arc of the equator, intercepted between the meridian of that place and the first meridian; the longitude, therefore, is the measure of the angle between the two meridians. The longitude is east or west, according as the place is situated on the right or on the left of the first meridian, when we look towards the north pole.

5. The difference of longitude between two places is the arc of the equator intercepted between the meridians of those places, or the measure of the angle which they include; hence, when the longitudes of the

places are of the same denomination, that is, either both east or both west, the difference is found by *subtracting* the one from the other; but when they are of contrary denominations the difference is found by adding the one to the other.

6. The latitude of a place is its distance from the equator, measured on the meridian of the place. Latitude, therefore, is north or south, according to the pole towards which it is measured, and cannot exceed 90° .

7. The small circles drawn parallel to the equator are called *parallels* of latitude. The arc of a meridian, intercepted between two such parallels, drawn through any two places, measures the difference of latitude of those places: when the latitudes are of the same denomination the difference of latitude is found by subtraction, but when the denominations are not the same the difference of latitude is found by addition, like difference of longitude.

8. The horizon of any place is an imaginary plane, conceived to touch the surface of the earth at that place, and to be extended to the heavens; such a plane is called the *sensible horizon*, and one parallel to it, but passing through the earth's centre is the *rational horizon* of the place. A line drawn across the horizon and through the place, in the plane of its meridian, is the meridian of the horizon, or the north and south line; the horizontal line through the same point, and perpendicular to this, is the east and west line. Besides the North, South, East, and West, points thus marked on the boundary of the horizon, this boundary is conceived to be subdivided into other intermediate points, corresponding to the divisions in the circle at the top of next page.

9. The *course* of a ship is the angle which her track makes with the meridians; so long as this angle remains the same, the ship is said to sail on the same *rhumb line*, or *loxodromic curve*. The magnitude of the angle or the course is indicated by the *mariner's compass*.

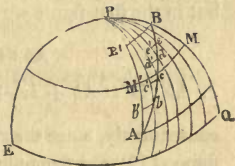
10. The *Mariner's compass* consists of a circular card, whose circumference is divided into thirty-two equal parts, called *points*, and each of these are subdivided into four equal parts, called quarter points; across this card is fixed a slender bar of magnetized steel, called the *needle*; the tapering extremities of which point to two diametrically opposite divisions of the card. These opposite divisions are marked N. and S., corresponding to the *north* and *south* poles, or ends, of the

which each measures the 120th of a nautical or geographical mile*, and as half a minute is the 120th of an hour, it follows that the number of knots, and parts of a knot, run in half a minute expresses the number of miles, and parts of a mile, run in an hour, at the same rate of sailing.

On Plane Sailing.

(64.) Let the annexed diagram represent a portion of the earth's surface, P being the pole, and EQ the equator.

Let AB be any rhumb line, or track described by a ship in sailing on a single course from A to B. Conceive the path of the ship to be divided into portions *Ab, bc, cd, &c.* so small that each may differ insensibly from a straight line, and draw meridians through these several divisions, as also the parallels of latitude *bb', cc', dd', &c.*;



we shall thus have a series of triangles described on the surface of the globe, but so small that each may be considered as a plane triangle. These triangles are all similar, for the angles at *b', c', d', &c.* are right-angles, and the ship's path cuts all the meridians at equal angles; hence (Geom. prop. 9, Book 6,)

$$Ab : Ab' :: bc : bc' :: cd : cd', \&c.$$

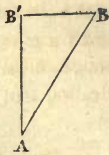
therefore, (Geom. prop. 5, Book 5,)

$$Ab : Ab' :: Ab + bc + cd + \&c. : Ab' + bc' + cd' + \&c.$$

But $Ab + bc + cd + \&c.$ is the whole distance sailed, and $Ab' + bc' + cd' + \&c. = AB'$, is the difference of latitude between A and B; consequently, if a right-angled triangle ABB' , similar to the small triangle Abb' , be constructed, that is, one in which the angle A is equal

* The geographical mile is one minute of the earth's circumference. Taking the diameter at 7916 English miles, the geographical mile will be about 6079 feet.

to the course, and if the hypotenuse AB represent the distance sailed, the side AB' will represent the difference of latitude. Moreover the other side BB', or that opposite to the course, will represent the sum $b'b + c'c + d'd + \&c.$ of all the minute *departures* which the ship makes from the successive meridians which it crosses; for as the triangle ABB', in this last diagram, is similar to the small triangle Abb', in the former, we have



$$Ab : bb' :: AB : BB' \dots (1);$$

but in the first figure we have

$$Ab : bb' :: bc : cc' :: cd : dd', \&c.$$

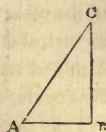
$$\therefore Ab : bb' :: Ab + bc' + cd + \&c. : bb' + cc' + dd' + \&c. \dots (2);$$

consequently, since the three first terms of (1) are respectively equal to those of (2), the remaining terms BB' , $bb' + cc' + dd' + \&c.$ must be equal. This last quantity is called the *departure* of the ship in sailing from A to B. It follows, therefore, that the *distance sailed*, the *difference of latitude* made, and the *departure*, are correctly represented by the *hypotenuse* and *sides* of a right-angled plane triangle, in which the angle opposite the departure is the *course*, so that when any two of these four things are given the others may be found simply by the resolution of a right-angled plane triangle; as far, therefore, as these particulars are concerned the results are the same as if the ship were sailing on a plane surface, the meridians being parallel straight lines, and the parallels of latitude cutting them at right-angles; and hence that part of Navigation in which only distance sailed, departure, difference of latitude, and course are considered, is called *Plane sailing*.

EXAMPLES.

1. A ship from latitude $47^{\circ} 30' N.$ has sailed S. W. by S. 98 miles. What latitude is she in, and what departure has she made?

Let C be the place sailed from, CB the meridian, the angle C = 3 points = $33^{\circ} 45'$, and CA = 98 miles, the distance sailed; then CB will be the difference of latitude, and BA the departure.



As rad.	10.	As rad.	10.
: Distance 98	1.9912261	: Dist.	1.9912261
: : cos. course $33^{\circ} 45'$	9.9198464	: : sin. course	9.7447390
: Diff. of lat. 81.48	1.9110725	: Departure 54.45	1.7359651

Latitude left $47^{\circ} 30' N.$

Diff. of lat. = 81.48 minutes = 1 22 S. Dep. = 54.45 miles W.

Latitude in $46^{\circ} 8' N.$

2. A ship sails for 24 hours on a direct course, from lat. $38^{\circ} 32' N.$, till she arrives at lat. $36^{\circ} 56' N.$; the course is between the S. and E., and the rate $5\frac{1}{2}$ miles an hour. Required the course, distance, and departure.

Lat. left $38^{\circ} 32' N.$

$24 \times 5\frac{1}{2} = 132$ miles, the distance.

Lat. in $36^{\circ} 56' N.$

Diff. 1 36 = 96 miles

As Dist.	132	2.1205739	As rad.	10.
: Rad.	10.		: Dist.	2.1205739
: : Diff. lat. 96	1.9822712		: : sin. course	9.8364771
: cos. course $43^{\circ} 20'$	9.8616973		: Dep. 90.58	1.9570510

Hence the course is S. $43^{\circ} 20' E.$, and the departure 90.58 miles E.

3. A ship sails from lat. $3^{\circ} 52' S.$ to lat. $4^{\circ} 30' N.$, the course being N. W. by W. $\frac{1}{2} W.$; required the distance and departure.

Distance 1065 miles, Departure 938.9 miles W.

4. Two ports lie under the same meridian, one in latitude $52^{\circ} 30' N.$,

and the other in latitude $47^{\circ} 10' N.$ A ship from the southernmost sails due east, at the rate of 9 miles an hour, and two days after meets a sloop which had sailed from the northernmost port; required the sloop's direct course and distance run.

Course S. $53^{\circ} 28' E.$, or S. E. $\frac{3}{4} E.$; distance run 537.6 miles.

5. If a ship from lat. $48^{\circ} 27' S.$, sail S. W. by W., 7 miles an hour, in what time will she arrive at the parallel of $50^{\circ} S.$?

In 23.914 hours.

6. If after a ship has sailed from lat. $40^{\circ} 21' N.$ to lat. $46^{\circ} 18' N.$, she be found 216 miles to the eastward of the port left; required her course and distance sailed.

Course N. $31^{\circ} 11' E.$, dist. 417.3 miles.

Traverse Sailing.

(65.) When a ship, in going from one place to another, sails on different courses, it is called *traverse sailing*; and the determination of the single course and distance from the one place to the other is called *working* or *compounding* the traverse. To effect this, it is obviously merely necessary to find the difference of latitude, and departure, due to each distinct course, to take the aggregate of these for the whole difference of latitude and departure, and from these to find, as in last article, the single course and distance. It is usual in thus compounding courses to form a table consisting of six columns, called a *traverse table*, and in the first column to register the several component courses, and against them, in the second column, the proper distances; the next two columns, marked N. and S., are to receive the several differences of latitude, whether N. or S., due to each course, and distance, and the two remaining columns marked E. and W. are to receive, in like manner, the corresponding *eastings* and *westings*, that is, the departures. When these several particulars are all inserted, the columns are added up, and the difference of the results of the N. and S. columns will be the required difference of latitude, and the difference of the results of the E. and W. columns will be the corresponding departure.

The columns appropriated to the differences of latitude and depar-

tures are usually filled up from a table already computed to every quarter point of the compass, and to all distances from one mile up to a hundred or 120; so that, by entering this table with any given course and distance, the proper difference of latitude and departure is found by inspection. Most books on navigation contain also a second and more enlarged traverse table, being computed to every course from a quarter of a degree up to forty-five degrees. This latter table we have not thought it necessary to insert in our collection, but the former we have given (Table iv.), and its use is fully explained in the introduction prefixed.

But there is another mode of finding the direct course and distance, much practised by seamen, viz. by construction. To facilitate this construction the *mariner's scale* is employed, which is a two-foot flat rule exhibiting several scales on each side, by help of which and a pair of compasses the usual problems in sailing may be all solved. One of these scales is a scale of chords, commonly called a scale of rhumbs, being confined to every quarter point of the compass; and another is a more enlarged scale of chords, being to every single degree. Both these scales are constructed in reference to the same common radius, so that the chords on the scale of rhumbs belong to that circle whose radius equals the chord of 60° on the scale of chords; and the method of laying down a traverse from these scales, and one of equal parts, and of thence measuring the equivalent single course, and distance made good, will be at once understood from the following examples.

EXAMPLES.

1. A ship sails from a place in lat. $24^\circ 32'$ N., and has run the following courses and distances, viz.

1st, S.W. by W., distance 45 miles; 2d, E.S.E., distance 50 miles; 3d, S.W., distance 30 miles; 4th, S.E. by E., distance 60 miles; 5th, S.W. by S. $\frac{1}{4}$ W., distance 63 miles: required her present latitude, with the direct course and distance from the place left to the place arrived at.

Traverse Table.

Courses.	Dist.	Difference of Lat.		Departure.	
		N.	S.	E.	W.
S.W. by W.	45		25.0		37.4
E.S.E.	50		19.1	46.2	
S.W.	30		21.2		21.2
S.E. by E.	60		33.3	49.9	
S.W. by S. $\frac{1}{4}$ W.	63		50.6		37.5
			149.2	96.1	96.1

It appears from the results of this table that the difference of latitude made by the ship during the traverse is $149.2 \text{ S.} = 2^{\circ} 29' \text{ S.}$

Lat. left $24^{\circ} 32' \text{ N.}$

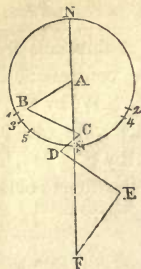
Diff. lat. $2 \text{ } 29 \text{ S.}$

Lat. in $22 \text{ } 3 \text{ N.}$

It appears also that the departures east are equal to the departures west, so that the ship has returned to the meridian she sailed from, consequently the direct course from the place left to that come to is due south, and the distance is equal to the difference of latitude, which is 149.2 miles.

The construction of this traverse is as follows.

With the chord of 60° , taken from the line of chords on the mariner's scale, describe the horizon circle, and draw the north and south line N.S. From the line of rhumbs take the chords of the several courses, and as these are all southerly they must be laid off from the south point S, those which are westerly to the left, and those which are easterly to the right, their extremities being marked 1, 2, 3, &c. in the order of the courses. This done, lay off from any convenient scale of equal parts, and in the direction A1 the distance AB sailed on the first course; then in the



direction parallel to A2, the distance BC sailed on the second course; in the direction parallel to A3, the distance CD on the third course; in the direction parallel to A4, the distance DE on the fourth course; and, lastly, in the direction parallel to A5, the distance EF on the third course; then F will represent the plane of the ship at the end of the traverse; FA, being applied to the scale of equal parts, will show the distance made good, and the chord of the arc included between this distance, and the meridian, being applied to the line of rhumbs, will show the direct course. In the present case the intercepted arc will be 0, showing that F is on the meridian of A.

2. A ship from Cape clear, in lat. $51^{\circ} 25' N.$, sails 1st, S.S.E. $\frac{1}{4}$ E., 16 miles; 2d, E.S.E., 23 miles; 3d, S.W. by W. $\frac{1}{2}$ W., 36 miles; 4th, W. $\frac{3}{4}$ N., 12 miles; 5th, S.E. by E. $\frac{1}{4}$ E., 41 miles: required the distance made good, the direct course, and the latitude in?

Traverse Table.

Courses.	Dist.	Difference of Lat.		Departure.	
		N.	S.	E.	W.
S.S.E. $\frac{1}{4}$ E.	16		14.5	6.3	
E.S.E.	23		8.8	21.3	
S.W. by W. $\frac{1}{2}$ W.	36		17.0		31.8
W. $\frac{3}{4}$ N.	12	1.8			11.9
S.E. by E. $\frac{1}{4}$ E.	41		21.1	35.2	
		1.8	61.4	63.3	43.7
			1.8	43.7	
			59.6	19.6	

Lat. left . . . $51^{\circ} 25' N.$

Diff. lat. $59.6m$. . . $1 \ 0 \ S.$

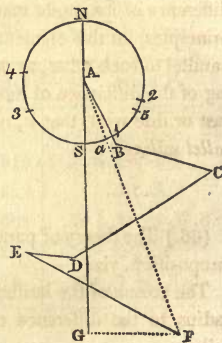
Lat. in . . . $50 \ 25 \ N.$

As diff. lat. $59^{\circ}6'$	1.7752463	As sin. course	9.4946205
: rad.	10	: departure	1.2922561
: : departure 19.6	1.2922561	: : rad.	10
: tan. course $18^{\circ}12'$	9.6170098	: distance 62.74	1.7976356

therefore, as the difference of latitude is south, and the departure east, the direct course is S. $18^{\circ}12'$ E., and the distance made good 62.74 miles.

To construct this traverse, describe, as before, the horizon circle, with a radius equal to the chord of 60° , and taking from the line of rhumbs the chord of the first course, $2\frac{1}{4}$ points, apply it from S. to 1, to the right of S.N., as this course is south-easterly; apply, in like manner, the chord of the second course, 6 points from S. to 2, also to the right of the meridian line; apply the chord of the third course, $5\frac{1}{2}$ points from S. to 3, to the left of the meridian, the chord of the fourth course, $7\frac{1}{4}$ from N. to 4, to the left of N.S., this course being north-westerly, and, lastly, apply the chord of the fifth course, $5\frac{1}{4}$ points, from S.

to 5, to the right of S.N. In the direction A1, lay off the distance AB = 16 miles from a scale of equal parts; in the direction parallel to AQ, lay off the distance BC = 23 miles; in the direction parallel to A3, lay off CD = 36; in the direction parallel to A4, lay off DE = 12 miles; and, lastly, in the direction parallel to A5, lay off EF = 41, then F will be the place of the ship at the end of the traverse; consequently, AF will be the distance made good, and the angle FAS the direct course;



applying, therefore, the distance AF to the scale of equal parts, we shall find it reach from 0 to 62 $\frac{3}{4}$; and applying the distance Sa to the line of chords, we shall find it reach from 0 to 18° .

3. A ship from lat. $28^{\circ}32'$ N., has run the following courses, viz. 1st, N.W. by N., 20 miles; 2d, S.W., 40 miles; 3d, N.E. by E., 60 miles; 4th, S.E., 55 miles; 5th, W. by S., 41 miles; 6th, E.N.E., 66 miles. Required her present latitude, the distance made good, and the direct course from the place left to that come to.

The direct course is due east, and distance 70·2 miles, the ship being in the same latitude at the end as at the beginning of the traverse.

4. A ship from lat. $41^{\circ} 12' N.$, sails S.W. by W., 21 miles; S.W. $\frac{1}{4}$ S. 31 miles; W.S.W. $\frac{1}{2}$ S., 16 miles; S. $\frac{3}{4}$ E., 18 miles; S.W. $\frac{1}{4}$ W., 14 miles; and W. $\frac{1}{2}$ N., 30 miles: required the latitude of the place arrived at, and the direct course and distance to it.

Lat. $40^{\circ} 5' N.$; course S. $52^{\circ} 49' W.$; distance 111·7 miles.

5. A ship runs the following courses, viz.

1st, S.E., 40 miles; 2d, N.E., 28 miles; 3d, S.W. by W., 52 miles; 4th, N.W. by W., 30 miles; 5th, S.S.E., 36 miles; 6th, S.E. by E., 58 miles: required the direct course and distance made good.

Direct course S. $25^{\circ} 59' E.$, or S.S.E. $\frac{1}{4}$ E. nearly; distance 95·87 miles.

These examples will, perhaps, suffice to illustrate the principles of plane sailing, in which, course, distance, difference of latitude, and departure, are the only things concerned. The determination of the difference of longitude made on any course cannot be effected by these principles, for this element is not the same as if the meridians were all parallel to each other, as is the case with the other elements. The finding of the difference of longitude is the easiest when the ship sails due east or due west, that is, upon a parallel of latitude; this is called *parallel sailing*.

Parallel Sailing.

(66.) The theory of parallel sailing is comprehended in the following proposition, viz.

The cosine of the latitude of the parallel is to the distance run as the radius to the difference of longitude. This may be demonstrated as follows.

In the figure, at page 66, let IQH represent the equator, and BDA any parallel of latitude; CI will be the radius of the equator, and cB the radius of the parallel. Let BD be the distance sailed, then the difference of longitude will be measured by the arc IQ of the equator, and since (Geom., prop. 12, Cor. 2, B. 7) similar arcs are to each other as the radii of the circles to which they belong, we have

$$cB : CI :: \text{dist. } BD : \text{diff. long. } IQ.$$

But cB is the cosine of the latitude IB to the radius CI , that is, cB is CI times the trigonometrical cosine of the latitude; hence the above proportion is

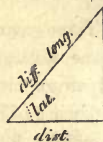
$$CI \times \cos. \text{lat.} : CI :: \text{distance} : \text{diff. long.}$$

$$\therefore \cos. \text{lat.} : \text{Rad. } (=1) :: \text{distance} : \text{diff. long.} \dots (1).$$

Corollary: hence if the distance between any two meridians, measured on a parallel in latitude L be D , and the distance of the same meridians, measured on a parallel, in latitude L' be D' , we shall have, (Geom., prop. 15, Cor. 2, Book 5,)

$$\cos. L : D :: \cos. L' : D' \dots (2).$$

Hence if one of the legs of a right-angled triangle represent the distance run on any parallel, and the adjacent acute-angle be equal to the degrees of lat. of that parallel, then the hypotenuse will represent the difference of longitude, since this hypotenuse will be determined by the foregoing proportion (1). It follows, therefore, that any problem in parallel sailing may be solved by the traverse table, computed to degrees, as a simple case of plane sailing; for by considering the latitude as the course, and the distance as the difference of latitude, the corresponding distance in the table will express the difference of longitude.



EXAMPLES.

1. A ship from latitude $53^\circ 56'$ N., longitude $10^\circ 18'$ E., has sailed due west, 236 miles: required her present longitude.

By the rule

As cos. lat.	$53^\circ 56'$	.	.	.	9.7699134
: radius	.	.	.	.	10.
:: distance	236	.	.	.	2.3729120
: diff. long.	408.87	.	.	.	<u>2.6029986</u>

$$\begin{array}{rcl}
 \text{Long. left} & . & 10^{\circ} 18' \text{ E.} \\
 \text{Diff. long.} = \frac{400}{60} \text{ degrees} = & & 6 \ 40 \ \text{W.} \\
 & & \hline
 \text{Long. in} & . & 3 \ 38 \ \text{E.} \\
 & & \hline
 \end{array}$$

2. If a ship sail E. 126 miles, from the North Cape, in lat. $71^{\circ} 10'$ N., and then due N., till she reaches lat. $73^{\circ} 26' \text{ N.}$; how far must she sail W. to reach the meridian of the North Cape?

Here the ship sails on two parallels of latitude, first on the parallel of $71^{\circ} 10'$, and then on the parallel of $73^{\circ} 26'$, and makes the same difference of longitude on each parallel. Hence, by the corollary,

$$\begin{array}{rcl}
 \text{As cos. lat. } 71^{\circ} 10' \text{ arith. comp. } 0.4910444 & & \\
 : \text{ distance } 126 & . & 2.1003705 \\
 :: \text{ cos. lat. } 73 \ 26 & . & 9.4550441 \\
 & & \hline
 : \text{ distance } 111.3 & . & 2.0464590. \\
 & & \hline
 \end{array}$$

3. A ship in latitude 32° N. sails due east, till her difference of longitude is 384 miles; required the distance run

325.6 miles.

4. If two ships in latitude $44^{\circ} 30' \text{ N.}$, distant from each other 216 miles, should both sail directly south till their distance is 256 miles, what latitude would they arrive at?

$32^{\circ} 17' \text{ S.}$

5. Two ships in the parallel of $47^{\circ} 54' \text{ N.}$, have $9^{\circ} 35'$ difference of longitude, and they both sail directly south, a distance of 836 miles: required their distance from each other at the parallel left, and at that reached.

385.5 miles, and 479.9 miles.

Middle Latitude Sailing.

(67.) Having seen how the longitude which a ship makes when sailing on a parallel of latitude may be determined, we come now to examine

the more general problem, viz. to find the longitude a ship makes when sailing upon any oblique rhumb.

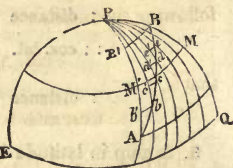
There are two methods of solving this problem, the one by what is called *middle latitude sailing*, and the other by *Mercator's sailing*. The first of these methods is confined in its application, and is moreover somewhat inaccurate even where applicable; the second is perfectly general, and rigorously true; but still there are cases in which it is advisable to employ the method of middle latitude sailing, in preference to that of Mercator's sailing; it is, therefore, proper that middle latitude sailing should be explained, especially since, by means of a correction to be hereafter noticed, the usual inaccuracy of this method may be rectified.

Middle latitude sailing proceeds on the supposition that the departure or sum of all the meridional distances $b'b$, $c'c$, $d'd$, &c. from A to B, is equal to the distance M'M of the meridians of A and B, measured on the middle parallel of latitude between A and B.

This supposition becomes very inaccurate when the course is small, and the distance run great; for it is plain that the middle latitude distance will receive a much greater accession than the departure, if the track of B cuts the successive meridians at a very small angle.

The principle approaches nearer to accuracy as the angle A of the course increases, because then as but little advance is made in latitude, the several component departures lie more in the immediate vicinity of the middle latitude parallel. But still, as in very high latitudes, a small advance in latitude makes a considerable difference in meridional distance, this principle is not to be recommended in such latitudes if much accuracy is required.

By means, however, of a small table of corrections, recently constructed by *Mr. Workman*, and judiciously introduced by *Mr. Riddle* in the second edition of his valuable *Treatise on Navigation*, the imperfections of the middle latitude method may be removed, and the results of it rendered in all cases accurate. This table we have given at the end of the present volume, and have explained its construction in the introductory explanation to the Tables.



The rules for middle latitude sailing may be thus deduced.

It has been seen at (64) that the difference of latitude, departure, and distance, sailed on any oblique rhumb, will be all accurately represented by the sides AB' , $B'B$, AB , of a plane triangle. Now, by the present hypothesis, the departure $B'B$ is equal to the middle latitude distance between the meridians of the places sailed from, and arrived at, so that the difference of longitude of the two places of the ship is the same as if it had sailed the distance $B'B$, on the middle latitude parallel; the determination of this difference of longitude is, therefore, reduced to a case of parallel sailing, for BB' , now representing the distance on the parallel, and an angle $A'BB'$ being made equal to the latitude of that parallel, we shall have the difference of longitude, represented by the hypotenuse $A'B$. We thus have the following theorems, viz. in the triangle $A'B'B$,



$$\cos. A'BB' : BB' :: \text{radius} : BA';$$

that is,

$$\text{I. Cos. mid. lat. : departure} :: \text{radius} : \text{diff. of long.}$$

In the triangle $A'BA$,

$$\sin. A' : AB :: \sin. A : A'B;$$

that is,

$$\text{II. Cos. mid. lat. : distance} :: \sin. \text{course} : \text{diff. long.}$$

In the triangles ABB' , $A'BB'$,

$$AB' \tan. A = B'B; A'B \cos. A'BB' = B'B;$$

therefore,

$$AB' : A'B :: \cos. A'BB' : \tan. A;$$

that is,

$$\text{III. Diff. lat. : diff. long.} :: \cos. \text{mid. lat.} : \tan. \text{course.}$$

These three proportions comprise the theory of middle latitude sailing, and when to the middle latitude the proper correction, taken from Mr. Workman's table, is added, these theorems will be rendered strictly accurate.

EXAMPLES.

1. A ship, in latitude $51^{\circ} 18' N.$, longitude $22^{\circ} 6' W.$, is bound to a place in the S.E. quarter, 1024 miles distant, and in lat. $37^{\circ} N.$: what is her direct course and distance, as also the difference of longitude between the two places;

$$\begin{array}{rcl}
 \text{Lat. from } 51^{\circ} 18' N. & \} & \text{Sum of latitudes} \quad . \quad 88^{\circ} 18' \\
 \text{Lat. to } 37 \quad 0 \quad N. & \} & \text{Mid. lat.} \quad . \quad . \quad 44 \quad 9 \\
 \hline
 \text{Diff. lat. } 14 \quad 18 & = & 858 \text{ miles.}
 \end{array}$$

For the course.		For the diff. long.	
As distance	1024 3.0103000	cos. mid. lat. $44^{\circ} 9'$ ar. com.	0.1441668
: radius	10.	: tan. course $33 \quad 5$	9.8138993
: : diff. lat.	858 2.9334873	: : diff. lat. 858	2.9334873
: cos. course $33^{\circ} 5'$	9.9231873	: diff. long. 779	2.8915534

In this operation the middle latitude has not been corrected, so that the difference of longitude here determined is not without error. To find the proper correction look for the given middle latitude, viz. $44^{\circ} 9'$ in the table of corrections, the nearest to which we find to be 45° ; against this and under 14° diff. of lat. we find $27'$, also under 15° we find $31'$, the difference between the two being $4'$; hence corresponding to $14^{\circ} 18'$ the correction will be about $28'$. Hence the corrected middle latitude is $44^{\circ} 37'$, therefore,

$$\begin{array}{rcl}
 \text{cos. corrected mid. lat. } 44^{\circ} 37' \text{ ar. comp.} & 0.1483780 \\
 : \text{ tan. course } 33 \quad 5 & 9.8138993 \\
 : : \text{ diff. lat. } 858 & 2.9334873 \\
 \hline
 : \text{ diff. long. } 783.6 & 2.8957646;
 \end{array}$$

therefore, the error in the former result is about $7\frac{1}{2}$ miles.

2. A ship sails in the N.W. quarter, 248 miles, till her departure is 135 miles, and her difference of longitude 310 miles: required her course, the latitude left, and the latitude come to.

Course N. $32^{\circ} 59'$ W.; lat. left $62^{\circ} 27'$ N.; lat. in $65^{\circ} 55'$ N.

3. A ship, from latitude 37° N., longitude $9^{\circ} 2'$ W., having sailed between the N. and W., 1027 miles, reckons that she has made 564 miles of departure; what was her direct course, and the latitude and longitude reached?

Course N. $33^{\circ} 19'$ W. or N.W. by N. nearly; lat. $51^{\circ} 18'$ N.; long. $22^{\circ} 8'$ W.

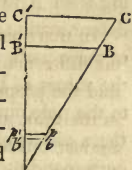
4. Required the course and distance from the east point of St. Michael's, lat. $37^{\circ} 48'$ N., long. $25^{\circ} 13'$ W., to the Start Point, lat. $50^{\circ} 13'$ N., long. $3^{\circ} 38'$, the middle latitude being corrected by Workman's Table.

Course N. $51^{\circ} 11'$ E.; distance 1189 miles.

Mercator's Sailing.

(68.) It has been already seen that when a ship sails on any oblique rhumb the difference of latitude, the departure, and the distance run, are truly represented by the sides of a right-angled plane triangle. The departure $B'B$ represents the sum of all the very small meridian distances, or elementary departures, $b'b$, $c'c$, &c. in the diagram, at page 124, the difference of latitude AB represents the sum of all the corresponding small difference in the figure referred to; and the distance AB , the sum of all the distances to which A these several departures and differences belong, and each of these elements are supposed to be taken so excessively small as to form on the sphere a series of triangles, differing insensibly from plane triangles.

Let $Ab'b$ in the annexed diagram represent one of these elementary triangles, $b'b$ will be one of the elements of the departure, and Ab' , the corresponding difference of latitude; and as $b'b$ is a small portion of a parallel of latitude, it will be to a similar portion of the equator, or of the meridian, as the cosine of its latitude to radius (66). This similar portion of the equator, or of the meridian, being the difference of



longitude between b' and b . Suppose now the distance Ab prolonged to p , till the departure $p'p$ is equal to the difference of longitude of b' , and b , then $b'b$ will be to $p'p$ as the cosine of the latitude of $b'b$ to the radius; but $b'b : p'p :: Ab' : Ap'$; hence the proper difference of latitude Ab' is to the increased difference Ap' as the cosine of the latitude of $b'b$ to the radius. Calling, therefore, the proper difference of latitude d , the increased difference D , the latitude of $b'b$, ℓ , and the radius R , we

have $D = \frac{Rd}{\cos. \ell} = Rd \sec. \ell$; the ship, therefore, having made the

small departure $b'b$, and the difference of latitude Ab' , must continue her course till the difference of latitude becomes D , in order that her departure may become equal to the difference of longitude corresponding to $b'b$. Conceiving all the elementary distances to be in this manner increased, the sum of all the corresponding increased departures will necessarily be the whole difference of longitude made by the ship during the course; to represent, therefore, the difference of longitude due to the departure $B'B$, and difference of latitude AB' , we must prolong AB' till AC' is equal to the sum of all the elementary differences increased as above, and the departure $C'C$, due to this difference of latitude, will represent the difference of longitude actually made in sailing from A to B . The determination of AC' requires the previous determination of all its elementary parts; if d be taken equal to $1'$ each of these parts will be expressed by $D = 1' \sec. \ell$, from which equation the values of D , corresponding to every minute of ℓ , from the equator to the pole, may be calculated; and by the continued addition of these there will be obtained, in succession, the values of the increased latitude corresponding to $1', 2', 3', \&c.$ of proper latitude; these values are called the *meridional parts*, corresponding to the several proper latitudes, and when registered in a table form a table of meridional parts, given in all books on Navigation.

The following may serve as a specimen of the manner in which such a table may be constructed, and, indeed, of the manner in which the first table of meridional parts was actually formed by *Mr. Wright*, the proposer of this ingenious and valuable method.

Mer. pts. of $1' = \text{nat. sec. } 1'.$

Mer. pts. of $2' = \text{nat. sec. } 1' + \text{nat. sec. } 2'.$

Mer. pts. of $3' = \text{nat. sec. } 1' + \text{nat. sec. } 2' + \text{nat. sec. } 3'.$

Mer. pts. of $4' = \text{nat. sec. } 1' + \text{nat. sec. } 2' + \text{nat. sec. } 3' + \text{nat. sec. } 4'.$

&c.

&c.

Hence, by means of a table of natural secants, we have

	Nat. secs.	Mer. parts.
Mer. pts. of $1' =$	1.0000000	$= 1.0000000$
Mer. pts. of $2' =$	$1.0000000 + 1.0000002$	$= 2.0000002$
Mer. pts. of $3' =$	$2.0000002 + 1.0000004$	$= 3.0000006$
Mer. pts. of $4' =$	$3.0000006 + 1.0000007$	$= 4.0000013$
&c.	&c.	

There are other methods of construction, but this is the most simple and obvious; we shall, however, presently have to advert to another process of computation, by which the meridional parts for any latitude may be found independently of previous calculations. The meridional parts, thus determined, are all expressed in geographical miles, because in the general expression $D = 1' \sec. l$, $1'$ is a geographical mile.

Having thus formed a table of meridional parts, (see Riddle's Navigation, or Robertson's Treatise,) if we enter it with the latitudes sailed from, and come to, and take the difference of the corresponding parts in the table, the remainder will be the meridional difference of latitude, or the line AC' in the preceding diagram, and the difference of longitude C'C will then be obtained by this proportion, viz.

1. *As radius is to the tangent of the course, so is the meridional difference of latitude to the difference of longitude;* or if the departure be given instead of the course then the proportion will be

2. *As the proper difference of latitude is to the departure, so is the meridional difference of latitude to the tangent of the course.* Other proportions immediately suggest themselves from the preceding figure.

(69.) As an example of Mercator's, or more properly of Wright's, sailing, let us take the following.

1. Required the course and distance from the east point of St. Michael's to the Start Point.

Start lat. $50^{\circ} 13' N.$ Mer. pts. 3494.8 long. $3^{\circ} 38' W.$

St. Michael's lat. $37^{\circ} 48' N.$ Mer. pts. 2453.1 long. $25^{\circ} 13' W.$

12 25 Mer. diff. lat. 0141.7 diff. long. 21 35 W.

60

60

Proper diff. lat. 745 miles.

1295 miles.

For the course.

As Mer. diff. lat. 1041.7 3.0177427

: radius . . . 10.

: : diff. long. 1295 3.1122698

: tan. course $51^{\circ} 11' E.$ 10.0945271

For the distance.

As cos. course . 9.7971501

: prop. diff. lat. 2.8721563

: : rad. . . 10.

: distance 1189 3.0750062.

(70.) In the absence of a table of meridional parts, a table of logarithmic tangents may be employed for the same purposes; and, indeed, the meridional parts corresponding to any given latitude may be expeditiously computed by help of such a table, and independently of any previous computations.

This method of computation will be found investigated at note C, at the end, in which we have mentioned one or two additional particulars respecting Mr. Wright's method. The practical rule is as follows, viz.

If the log. tangent of half the complement of any latitude be subtracted from 10, and the remainder be multiplied by 7915.7044679, &c. the product will give the meridional parts in miles, corresponding to that latitude.

From this rule the method of operating with logarithmic tangents, instead of with meridional parts, may be easily derived. Call t, t' the logarithmic tangents of the half complements of the latitudes left and reached, and put a for the constant multiplier 7915.7044, &c. Then, by the rule just given, the meridional difference of latitude will be

$$a \{ (10 - t') - (10 - t) \} = a(t - t') = (t - t') 10000 \div \frac{10000}{a}.$$

Now $\log. \frac{10000}{a} = .1015104$, therefore, the logarithm of the meridional difference of latitude is found by removing the decimal point in the difference $t - t'$ four places to the right, and then subtracting the constant number .1015103. Hence, if instead of the logarithm of the radius 10, we use 10.1015104, and instead of the meridional parts the logarithmic tangents t, t' , of the complements of the half latitudes, taking care in setting down the difference of these to remove the decimal point four places to the right, the proportion (1), at page 129, may be still employed. Thus, taking the foregoing example, the operation by this method will be as follows.

$$\begin{array}{rcl} \frac{1}{2} \text{ Co. lat. } \left\{ \begin{array}{l} \text{St. Michael's } 26^{\circ} 6' \quad \therefore t = 9.6901030 \\ \text{Start} \quad 19 \ 53\frac{1}{2} \quad \therefore t' = 9.5585051 \end{array} \right. \\ \hline t - t' = 1315.979 \\ \hline \text{As } t - t' \quad 1315.979 \text{ arith. comp.} \quad . \quad 6.8807448 \\ : \text{ Const. log.} \quad . \quad . \quad . \quad 10.1015104 \\ : : \text{ diff. long.} \quad 1295 \quad . \quad . \quad 3.1122698 \\ \hline : \text{ tan. course N. } 51^{\circ} 11' \text{ E.} \quad . \quad 10.0945250 \\ \hline \end{array}$$

The reason why the resulting logarithm here does not exactly coincide with that obtained by using the meridional parts, is that the meridional parts have been computed to but one place of decimals; if they had been computed to two or three places, the two results would have been exactly the same.

2. Given the Lizard in lat. $49^{\circ} 55' \text{ N.}$ Barbadoes in lat. $13^{\circ} 10' \text{ N.}$, and their difference of longitude 53° , or $3180' \text{ W.}$; to determine the course and distance.

Course S. $49^{\circ} 59' \text{ W.}$; distance 3429 miles.

3. A ship sails from lat. 37° N. long., $22^{\circ} 56' \text{ W.}$, on the course N., $33^{\circ} 19' \text{ E.}$, till she arrives at lat. $51^{\circ} 18' \text{ N.}$: required the distance sailed, and the longitude arrived at.

Distance 1027 miles; longitude in $9^{\circ} 45' \text{ W.}$

We shall here terminate the present chapter on the principles of Navigation, having now discussed the several cases of sailing which actually occur in practice. But the student who is desirous of prosecuting his inquiries on this very important branch of practical science to greater extent, will, of course, consult works expressly devoted to the subject. Of these the most elaborate in our language is the valuable "Elements" of Robertson, in two octavo volumes. The Treatise of Mr. Riddle is also an excellent work, abounding with practical examples very accurately solved, and upon the whole better adapted to modern practice, as well as more compendious, than Robertson's. Mr. Norie's Navigation is also a good practical book, and so is Dr. Bowditch's edition of "Hamilton Moore."

CHAPTER II.

APPLICATION OF SPHERICAL TRIGONOMETRY TO ASTRONOMICAL PROBLEMS.

(71.) The solution of Astronomical Problems forms one of the most useful and agreeable applications of the theory of spherical Trigonometry. To such inquiries the theory itself, no doubt, owes its origin, as well as many of the successive improvements which it has gradually received, so that a specimen of its use in the solution of astronomical problems may reasonably be looked for in a book on Trigonometry.

For the purpose of measuring the *angular* distances of the heavenly bodies from each other, and from the horizon, it is convenient to suppose them all situated as they really appear to an observer on the earth, viz. in a spherical concave surrounding our earth and concentric with it. This imaginary concave is called the celestial sphere, or the apparent heavens; in it all the apparent motions of the heavenly bodies are, for the convenience of trigonometrical application, supposed actually to take place; and the entire celestial sphere to revolve daily round the

earth as if this were at rest in its centre. All this is allowable, because the applications of which we speak are not affected by the enquiry; whether the motions which the heavenly bodies present to an observer on the earth are really as they appear or not.

At the opening of last chapter we defined several lines which geographers had found it convenient to consider as described on the surface of the earth; most of these astronomers extend to the heavens. Thus the plane of the earth's equator, when extended to the heavens, marks on the celestial sphere the great circle called the *equinoctial*, and in like manner, the meridians being extended to the heavens, mark out the celestial meridians; also the axis of the earth, about which its real motion takes place, when extended to the heavens, is the axis about which the apparent motion of the celestial sphere takes place: this axis marks out the north and south poles of the heavens.

As the sun performs its apparent revolution about the earth in 24 hours it passes over 15° in an hour; if then we consider, as astronomers do, that the day at any place commences at noon, or when the sun is on the meridian of that place, the time shown by the sun in any position will be expressed in degrees by the arc of the equinoctial, intercepted between the fixed meridian of the place, and that passing through the sun, or it will be expressed by the angle included by these meridians. Celestial meridians are, therefore, also called *hour circles*, and the angle between the meridian of the place and that through the sun is called the hour angle, or the *horary angle*. That meridian which is at right-angles to the meridian of the place is the six o'clock hour circle, since the sun obviously reaches it when half way between noon and midnight.

Besides these lines, thus transferred from the earth to the heavens, there are others peculiar to the celestial sphere, which must be mentioned; these are, 1st, the *ecliptic*, which is the great circle path described by the sun among the fixed stars in its apparent annual motion about the earth: in reality it is the path of the earth moving in a contrary direction about the sun. This circle crosses the equinoctial at an angle subject to an exceedingly small variation, determinable by observation and computation; its inclination to the equinoctial is about $23^{\circ} 28'$, but it is always given with the minutest attainable accuracy in the *Nautical Almanack*. The points where the ecliptic crosses the equinoctial are called the

equinoctial points: the sun enters these points about the 21st of March and the 23d of September; the former being called the *vernal equinox*, and the latter the *autumnal equinox*. These names are given because at such times the nights are equal in length to the days all over the world; for as the two poles of the earth are at these times symmetrically situated with respect to the sun, the circular boundary, which separates the enlightened hemisphere from the darkened, must pass through both poles; and hence any point on the earth will be as long in being carried, by the earth's uniform rotation, through the enlightened part as through the dark part.

The meridian through the equinoctial points is called the *equinoctial colure*.

The position of any point on the celestial sphere, like the position of a point on the terrestrial sphere, is marked out by its latitude and longitude. On the celestial sphere the circle of longitude is the ecliptic; and perpendiculars to this, passing, therefore, through the poles of the ecliptic, are the circles of celestial latitude; the point from which longitude is measured is the vernal equinoctial point. Commencing at this point, too, the ecliptic is divided into twelve parts, called signs; a sign is therefore 30° . The twelve signs are named, and symbolically expressed, as follow:

1. ♈ Aries.	4. ♋ Cancer.	7. ♎ Libra.	10. ♐ Capricornus.
2. ♉ Taurus.	5. ♌ Leo.	8. ♏ Scorpio.	11. ♒ Aquarius.
3. ♊ Gemini.	6. ♍ Virgo.	9. ♏ Sagittarius.	12. ♓ Pisces.

The first six of these signs are on the north of the equinoctial, the others on the south, and the vernal equinoctial point is called the first point of Aries. The longitude is measured from this point in but one direction, viz. in the order of the signs.

Besides the above method of marking out the position of a celestial body, by means of its latitude and longitude, there is another way, viz. by means of its *Right Ascension* and *Declination*. The right ascension is measured on the equinoctial from the first of Aries, in the order of the signs, and the declination is measured on the perpendicular to this, or circle of declination passing through the object. We see, therefore, that what on the terrestrial sphere is latitude and longitude, is on the celestial

sphere declination and right ascension; and parallels of latitude on the one correspond to parallels of declination on the other. Of these the two which are $23^{\circ} 28'$ from the equinoctial, one on each side, and which therefore touch the ecliptic in the first points of Cancer and Capricorn, are called the *tropics* of Cancer and of Capricorn. These first points of Cancer and Capricorn are respectively called the summer and winter *solstice*; because for a day or two before and after the sun enters them he appears to be stationary, and the days to be of equal length, so slowly does his declination at those times change, for his motion is obviously very nearly parallel to the equinoctial. The meridian, through the solstitial points, is called the *solstitial colure*, and that through the equinoctial points, the *equinoctial colure*.

Having described the principal circles and points of the celestial sphere which are considered as permanent, or which do not alter with the situation of the observer on the earth, we come now to describe those which change with his place. The principal of these is the *horizon*, which has been defined already (63), and *vertical circles* which are perpendicular to the horizon, and on which the altitudes of celestial objects are measured.

These vertical circles all meet in two points diametrically opposite, viz. the poles of the horizon; that one which is directly over the head of the observer is called his *zenith*, and the opposite one his *nadir*. That vertical which passes through the east and west points of the horizon is called the *prime vertical*; it necessarily intersects the meridian of the place (which passes through the north and south points) at right-angles.

The *azimuth* of a celestial object is an arc of the horizon, comprised between the meridian of the observer and the vertical circle through the object, and hence vertical circles are sometimes called azimuth circles.

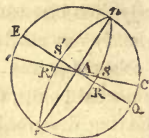
The *amplitude* of a celestial object is the arc of the horizon, comprised between the east point and the point where the object rises, or between the west point and that where it sets; the one is called the rising amplitude, the other the setting amplitude.

These definitions and remarks will suffice to render the following problems intelligible.

PROBLEM I.

(72.) Given the sun's right ascension and declination to determine his longitude and the obliquity of the ecliptic.

Let $nEsQ$ represent the celestial meridian through the first of Cancer and Capricorn, that is, let it be the solstitial colure, ns the axis of the sphere, EQ the equator, eC the ecliptic, and nSs the declination circle, passing through the sun S ; then ARS is a right-angle, and in the right-angled spherical triangle ARS there are given the right ascension AR , and the declination RS to find the longitude AS , and the obliquity SAR , which is an easy operation in right-angled spherics. It is necessary, however, to remark that as celestial longitude and right ascension are measured from A , the first point of Aries in the direction AS of the signs quite round the celestial sphere, when, of the four quantities in the problem, the obliquity and the declination are given to find the others, we must know on what side of the equinoctial the sun is, that is, whether the declination is north or south, for if the sun have the north declination RS , the longitude will be AS ; but if it have the equal south declination $R'S'$, the longitude being measured in the direction ASC round the globe to S' , will be, instead of $A'S'$, $360^\circ - A'S'$.



It is moreover necessary to know not only on which side of the equinoctial the sun is, but also on which side of the tropic; for the sun, in passing from a tropic to the equinox, descends through the same gradations of declination as it ascended through in passing from the preceding equinox to the tropic, although its longitude and right ascension goes on increasing; in addition, therefore, to knowing whether the declination is north or south, we must also know whether it be encreasing or decreasing, in order to determine the longitude and right ascension without ambiguity; and these particulars will be known from knowing the time of the year when the proposed declination is observed; thus from the 21st of March to the 21st of June, during which time the sun is in the first quadrant of the ecliptic, the sun's declination is north and increasing; it afterwards continues to decrease, still remaining north,

during the second quadrant, that is, till the 23d of September, from which, till the 21st of December, that is, during the third quadrant, the declination is south and increasing, after which, or during the fourth quadrant, the declination, still south, decreases till the 21st of March.

EXAMPLES.

1. Given the sun's right ascension on the 17th of May, $53^{\circ} 38'$, and its declination $19^{\circ} 15' 57''$: required his longitude and the obliquity of the ecliptic.

Applying Napier's rule to the right-angled triangle ARS, we have

$$\text{Rad.} \times \cos. AS = \cos. AR \cos. RS$$

$$\text{Rad. sin. AR} = \tan. RS \cot. A \therefore \cot. A = \frac{\text{Rad. sin. AR}}{\tan. RS}.$$

Hence the computation for AS and A is as follows.

For the longitude AS.			For the obliquity A.	
cos. AR	$53^{\circ} 38' 0''$	9.7730185	sin. AR	9.9059247
cos. RS	$19 15 57$	9.9749710	tan. RS arith. comp.	0.4565209
cos. AS	$55 57 43$	9.7479895	cot. A $23^{\circ} 27' 50\frac{1}{2}''$	10.3624456

2. On the 31st of March, 1816, the sun's declination was observed at Greenwich to be $4^{\circ} 13' 31\frac{1}{2}''$: required his right ascension, the obliquity of the ecliptic being $23^{\circ} 27' 51''$.

The right ascension was $9^{\circ} 47' 59''$.

3. Required the sun's longitude on the 28th of November, 1810, when his declination was $21^{\circ} 16' 4''$, and his right ascension, in time, $16^h 14^m 58.4^s$, or in degrees $243^{\circ} 44' 36''$.

The longitude was $245^{\circ} 39' 10''$, or 8 signs $5^{\circ} 39' 10''$.

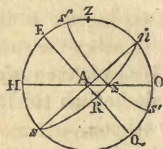
4. The sun's longitude being $8^{\circ} 7' 40' 56''$, and the obliquity of the ecliptic $23^{\circ} 27' 42\frac{1}{4}''$: required his right ascension in time.

The right ascension is $16^h 23^m 34^s$.

PROBLEM II.

Given the sun's declination to find the time of his rising and setting at any place whose latitude is known.

Let $nEsQ$ represent the meridian of the place, Z being the zenith, and HO the horizon, and let $s's''$ be the apparent path of the sun on the proposed day, cutting the horizon in S . Then the arc EZ will be the latitude of the place, and consequently EH , or its equal QO , will be the colatitude, and this measures the angle OAQ ; also RS will be the sun's declination, and AR , expressed in time, will express the time of sun rise from 6 o'clock, for nAs is the six o'clock hour circle.



Hence, in the right-angled triangle ARS , we have given RS and the opposite angle A to find AR , the time from 6 o'clock.

EXAMPLE I.

Required the time of sun rise at latitude $52^{\circ} 13' N$, when the sun's declination is $23^{\circ} 28'$.

By Napier's rule,

$$\text{Rad. sin. } AR = \cot. A \tan. RS = \tan. \text{lat.} \tan. \text{dec.}$$

$$\tan. 23^{\circ} 28' \quad . \quad . \quad 9.6376106$$

$$\tan. 52 \ 13 \quad . \quad . \quad 10.1105786$$

$$\sin. 34 \ 3 \ 21\frac{1}{4}'' \quad . \quad . \quad 9.7481892$$

4*

$$AR \text{ in time } 2^h \ 16' \ 13'' \ 25'''$$

6

$$3 \ 43 \ 46 \ 35 = \text{time of rising.}$$

* Degrees are converted into hours by multiplying by 4 and dividing by 60.

SCHOLIUM.

It should be here remarked that the time thus determined is *apparent time*, which is that which would be shown by a clock so adjusted as to pass over 24 hours during one apparent revolution of the sun, or from its leaving the meridian to its return to it again, the index pointing to 12, when the sun is on the meridian. But it is impossible that any clock can be so adjusted, because the interval between the successive return of the sun to the meridian is continually varying, on account of the unequal motion of the sun in its orbit, and of the obliquity of the ecliptic; each of these varying intervals is called a *true solar day*, and it is the mean of these during the year which is measured by the 24 hours of a well regulated clock, this period of time being a mean solar day; hence, at certain periods of the year, the sun will arrive at the meridian before the clock points to 12, and at other periods the clock will precede the sun; the small interval between the arrival of the index of the clock to 12 and of the sun to the meridian is called the *equation of time*, and it is given in page ii. of the Nautical Almanack for every day in the year; this correction, therefore, must always be applied to the apparent time determined by trigonometrical calculation to obtain the *true time*, or that shown by a well regulated clock or chronometer.

Another circumstance too must be taken into account, in order to determine the *apparent time* with rigorous accuracy, viz. the change in the declination of the sun from sunrise to noon. In the Nautical Almanack the declination of the sun is given for every day at noon, and if this be used in the computation we shall assume that the declination has not varied from sunrise to noon, which is not the case; hence it will be necessary to compute the declination for the time of sunrise, as determined above, and then to resolve the problem with this corrected declination. The correction is obtained by taking from the Nautical Almanack the variation of declination in 24 hours, and then finding by proportion the variation for the time required.

2. Required the time of sunrise at latitude $57^{\circ} 2' 54''$, when the sun's declination is $23^{\circ} 28'$?

3h 11^m 49^s.

2. At latitude $62^{\circ} 12' N.$ the altitude of the sun at 6 o'clock in the morning was found to be $18^{\circ} 20' 23''$; required his declination and azimuth.

Declination $20^{\circ} 50' 12'' N.$, Azimuth $79^{\circ} 56' 11''$ from N.

3. On the 20th of November, 1822, the declination of Aldebaran was $16^{\circ} 8' 36'' N.$, what was its altitude and azimuth when on the six o'clock hour circle of Greenwich, lat. $51^{\circ} 28' 40'' N.$?

Altitude $12^{\circ} 32' 3''$, Azimuth $79^{\circ} 46' 50''$ from N.

12 33 50 $\frac{1}{2}$?

PROBLEM IV.

The latitude of the place and the declination of the sun being given to find the time when it is due east, or upon the prime vertical, and the altitude at that time.

Having drawn the meridian of the plane as before, the vertical circle ZAN, at right-angles to it, will be the prime vertical, A being the east point of the horizon HAO: also P being the elevated pole, and S the place of the sun, ZP will be the colatitude, PS the codeclination, ZS the coaltitude, and ZPS the hour angle, or time from noon;



hence, in the right-angled spherical triangle SZP, there are given SP and PZ, to find SZ and the angle P. If the declination is not of the same name as the latitude, the sun will arrive at the prime vertical at S' before it rises: in this case the declination is to be considered as negative.

By Napier's rule

$$\text{Rad. cos. } P = \cot. SP \tan. PZ = \tan. \text{dec. cot. lat.}$$

$$\text{Rad. cos. } SP = \cos. SZ \cos. PZ \therefore \sin. \text{alt.} = \frac{\text{Rad. sin. dec.}}{\sin. \text{lat.}}$$

EXAMPLES.

1. On the 1st of August, 1831, the sun's declination was $18^{\circ} 10' 22''$ N., at what hour was he due east at Greenwich, and what was his altitude at that time?

For the hour angle.				For the altitude.			
tan. dec.	$18^{\circ} 10' 22''$	9.5162138		sin. dec.	.	9.4939924	
cot. lat.	51 28 40	9.9009509		sin. lat.	.	9.8934103	
cos. hor. angle	74 51 7	9.4171647		sin. alt. $23^{\circ} 29' 37''$		9.6005821	
	4						
	$4^h 59^m 24^s 28^t$						
	12						
	7 0 35 32.						

Hence the time is 35 seconds and a half past 7 o'clock, and the altitude $23^{\circ} 29' 37''$.

2. Given the sun's declination $5^{\circ} 8' 26''$ N., and his altitude when due east $16^{\circ} 53' 10''$; required the latitude of the place.

Latitude $17^{\circ} 58' \text{ N.}$

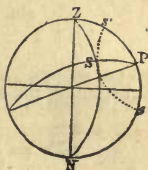
3. If the declination of a celestial object be $18^{\circ} 4' \text{ S.}$, what is its altitude when on the prime vertical of latitude $27^{\circ} 42' \text{ S.}$, and its distance from the meridian in time?

Altitude $41^{\circ} 51'$; Merid. distance in time $3^h 26^m 20^s$.

PROBLEM V.

To find the time when the apparent motion of a celestial object is perpendicular to the horizon, from having its declination and the latitude of the place given.

Let $s s'$ represent the parallel of declination, or the apparent diurnal path of the body, and let the vertical ZSN be drawn to touch it in S ; then S will be the place of the body when its apparent motion is in the direction SZ of the vertical, and therefore perpendicular to the horizon: through S draw the hour circle PS , which, being the shortest distance from P to ZN , is perpendicular to it (p. 85); hence the triangle PSZ is right-angled at S , and in which we have given the colatitude ZP , and the codeclination PS , to find the hour angle ZPS .

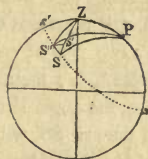


It is obvious that this problem will be impossible when Ps' exceeds PZ ; that is, when the declination is less than the latitude.

PROBLEM VI.

To determine upon what vertical a celestial object must be, in order that a small error, committed in taking its altitude, may have the least possible effect upon the hour angle.

Let S be the place of the sun, or other body, but by an error in taking its altitude let it be referred to S' . Draw $S' S''$ parallel to the horizon, and meeting the parallel of declination $s s'$ in S'' , then when the body is at S'' it will really have the coaltitude $ZS'' = ZS'$, which it was erroneously supposed to have at S , so that in the determination of the hour angle P , from the colatitude, the coaltitude, and the codeclination, the small angle $S' P S''$ will be the amount of the error.



As the triangle $SS' S''$ is, of course, exceedingly small, it may be regarded as a rectilinear triangle, right-angled at S' ; therefore $SS' = SS'' \sin. S''$, and $SS'' = \sin. PS \cdot \angle SPS''$, (see p. 121,) consequently,

$$SS' = \sin. S'' \sin. PS \cdot \angle SPS'' \therefore SPS'' = \frac{SS'}{\sin. S'' \sin. PS} \dots (1).$$

Now the angle S'' is equal to the angle ZSP , because $S' S''$ is the complement of each, and therefore, by the relation between the sides and angles of a spherical triangle, we have

$$\sin. S'' : \sin. PZ :: \sin. SZP : \sin. PS$$

$$\therefore \sin. S'' \sin. PS = \sin. PZ \sin. SZP.$$

Substituting the second member of this equation in (1), we have

$$SPS' = \frac{SS'}{\sin. PZ \sin. SZP} = \frac{\text{error in alt.}}{\cos. \text{lat.} \sin. \text{azimuth}}.$$

This expression will obviously be the least possible when the sine of the azimuth is the greatest possible, or when the azimuth is 90° ; that is, when the body is on the prime vertical.

Hence, in deducing the time from an altitude of any celestial body, it will be best to make the observation when the body is either exactly, or nearly, due east or due west.

PROBLEM VII.

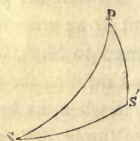
The latitudes and longitudes of two celestial objects being given, to determine their distance apart.

Let P represent the pole of the ecliptic, and PS, PS', two arcs of celestial latitude, drawn to the two objects S, S'; then will these arcs represent the colatitudes, the angle P will be the difference of longitude, and the arc SS' will be the distance sought, so that we have two sides and their included angle given to find the third side. In order to this we must first determine, agreeably to the method explained at page 99, a subsidiary angle ω , by the equation

$$\cot. \omega = \tan. PS \cos. P;$$

after which the side SS' is found by the equation

$$\cos. SS' = \frac{\cos. PS \sin. (\omega + PS')}{\sin. \omega}.$$



EXAMPLES.

1. Required the distance between Procyon and Capella, the latitude of Procyon being $15^{\circ} 58' 14''$ S., and its longitude $3^{\circ} 22' 55' 42''$; also the latitude of Capella being $22^{\circ} 51' 57''$ N., and longitude $2^{\circ} 18' 57' 57''$?

Taking the difference of the longitudes, we have for the angle P, $P = 33^{\circ} 57' 45''$; and for the colatitudes we have $PS = 105^{\circ} 58' 14''$, $PS' = 67^{\circ} 8' 3''$; hence the logarithmic process will be as follows:

tan. PS	105° 58' 14"	.	10.5433466	cos. PS	.	9.4395590
cos. P	33 57 45	.	9.9187658	sin. ω , ar. comp.		0.4865396
cot. ω	160 57 46	.	10.4621124	sin. ($\omega + PS'$)		9.8717340
<u>PS' = 67 8 3</u>				cos. SS' 51° 6' 39"	.	<u>9.7978326</u>
<u>$\omega + PS' = 228 5 49.$</u>						

In this example cot. ω is negative, because tan. PS is negative, and cos. P positive; also cos. SS' is positive, because cos. PS is negative, sin. ($\omega + PS'$) negative, and sin. ω positive. The operation will obviously be similar, when, instead of the latitudes and longitudes, the right ascensions and declinations of the two bodies are given to find their distance apart.

2. The latitude and longitude of a star S. are $38^{\circ} 40' 26''$ N., and $3^{\circ} 2' 4' 40''$; and of a star S', $13^{\circ} 26' 11''$ N., and $9^{\circ} 11' 41' 26''$; required their angular distance apart.

Distance $127^{\circ} 7' 11''$.

3. What is the distance between Sirius and Procyon, the right ascension of Sirius being $99^{\circ} 0' 21''$, and its declination $16^{\circ} 26' 35''$ S.; and the right ascension of Procyon $112^{\circ} 6' 47''$, and its declination $5^{\circ} 45' 3''$ N.

Distance $25^{\circ} 42' 10''$.

PROBLEM VIII.

Given the latitude of the place and the sun's declination to find the beginning and end of twilight.

Twilight commences in the morning and ends in the evening, when the sun is about 18° below the horizon. Hence, if PZ (see the diagram to next problem,) represent that portion of the meridian which is intercepted between the elevated pole and the zenith, and S' be that point in the sun's apparent path on any day which is 108° from Z, S' will be the place of the sun at the commencement of morning twilight, or at the termination of evening twilight; also PS' will be the codeclination, and PZ the colatitude; we thus have the three sides of the triangle PS'Z, to find the angle P. Hence, calling the sum of the three sides S, the formula for computing the hour angle P will be

$$\sin. \frac{1}{2} P = \sqrt{\frac{\sin. (\frac{1}{2} S - ZP) \sin. (\frac{1}{2} S - PS')}{\sin. ZP \sin. PS'}}$$

which is the same as

$$\sin. \frac{1}{2} P = \sqrt{\frac{\sin. \frac{1}{2} (\text{lat.} + 18^\circ + \text{codec.}) \sin. \frac{1}{2} (\text{dec.} + 18^\circ + \text{colat.})}{\cos. \text{lat.} \cos. \text{dec.}}}$$

a very convenient form for computation.

EXAMPLES.

1. At what time did twilight commence at Edinburgh, lat. $55^\circ 57' 20''$ N., on the 20th of August, 1831, when the sun's declination was $12^\circ 38' 9''$ N.?

lat. $+18^{\circ}$ $73^{\circ}57'20''$
 codec. $77\ 21\ 51$ ar. comp. sin. 0.0106480

2) $151\ 19\ 11$

$75\ 39\ 35\frac{1}{2}$ sin. 9.9862531

dec. $+18^{\circ}$ $30\ 38\ 9$

colat. $34\ 2\ 40$ ar. comp. sin. 0.2519393

2) $64\ 40\ 49$

$32\ 20\ 24\frac{1}{2}$ sin. 9.7283086

2) 19.9771490

sin. P $76^{\circ}54'54\frac{1}{2}''$ $9.9885745.$

Hence $P = 153^{\circ}49'49'' = (\text{in time}) 10^h 15^m 19\frac{1}{4}s$, so that twilight commenced in the morning at $1^h 44^m 40\frac{3}{4}s$, and ended in the evening at $10^h 15^m 19\frac{1}{4}s$.

2. At what time does the twilight begin at latitude $48^{\circ}38'56''$ N., when the sun's declination is $8^{\circ}28'54''$ N.?

Twilight begins at $3^h 20^m$.

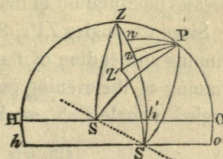
3. At what time does twilight end at latitude $52^{\circ}12'35''$ N., when the sun's declination is $15^{\circ}55'25''$ N.?

Twilight ends at $10^h 12\frac{1}{2}^m$.

PROBLEM IX.

Given the latitude of the place to determine on what day of the year the twilight is the shortest, and its duration on that day.

Let HO represent the horizon, and ho the parallel to it, 18° below it; also let PS be the declination circle, passing through the sun at sunset, and PZ, that passing through the zenith. Conceiving these two circles to revolve with S, PS will come to PS' when S comes to S' , and PZ will take some determi-



nate position PZ' . Now, since the angles ZPS , $Z'PS'$, are equal, we have, by taking from each the common part $Z'PS$, $ZPZ' = SPS'$; but SPS' , converted into time, expresses the duration of twilight, ZPZ' is therefore the least possible when the twilight is the shortest possible. Now since the sides PZ , PZ' , are both given, the side ZZ' will be the shortest when the opposite angle, P , is the least; (see equa. (A) p. 74,) hence when ZZ' is the shortest, the twilight is the shortest; but as the two sides $Z'S'$, ZS , of the triangle $ZZ'S'$, are given, the third side will be shortest when the angle S' is the least possible, and this is the case when Z' falls on ZS' , for then the angle is 0. Hence the twilight is shortest when the angle PSZ is equal to the angle $PS'Z$.

Let then z be the proper position of Z' ; we shall have $Zz = Zh' - zh' = zS' - zh' = h'S' = 18^\circ$, and because $PZ = Pz$, the arc Pn , bisecting the angle ZPz , will also bisect the base Zz , and be perpendicular to it (54); consequently,

$$\sin. ZPn = \sin. \frac{1}{2} SPS' = \frac{\sin. Zn}{\sin. PZ} = \frac{\sin. 9^\circ}{\cos. \text{lat.}} \dots (1);$$

also

$$\cos. Pn = \frac{\cos. PZ}{\cos. Zn};$$

and, in the right-angled triangle PnS' ,

$$\cos. PS' = \cos. n S' \cos. Pn = \frac{\cos. n S' \cos. PZ}{\cos. Zn};$$

that is,

$$\sin. \text{dec.} = \frac{\cos. 99^\circ \sin. \text{lat.}}{\cos. 9^\circ} = - \frac{\sin. 9^\circ \sin. \text{lat.}}{\cos. 9^\circ} = - \tan. 9^\circ \sin. \text{lat.} (2).$$

The declination being known by this equation, the day of shortest twilight is also known, (*Naut. Alm.*) The declination will be of a contrary name with the latitude as its sine is negative. Equation (1) expresses the duration of the twilight.

Since the angles ZPz , SPS' , are equal, the hour angles for the beginning and ending of the morning twilight, or for the ending and beginning of the evening twilight, are ZPS' , zPS' . Now, in the right-angled triangle PnS' , we have

$$\sin. n PS' = \frac{\sin. S'n}{\sin. PS'} = \frac{\sin. 99^\circ}{\cos. \text{dec.}} = \frac{\cos. 9^\circ}{\cos. \text{dec.}} \dots (3).$$

The sum of (1) and (3) gives the angle ZPS' , and their difference the angle $zPS' = ZPS$, and thus we have the hour angles for the beginning and end of the twilight.

EXAMPLES.

1. Required the time and duration of shortest twilight at Greenwich, lat. $51^{\circ} 28' 40''$, in the year 1832.

For the day.				For the duration.			
tan.	9°	.	9.1997125	sin.	9°	.	9.1943324
sin.	$51^{\circ} 28' 40''$		9.8934103	cos.	$51^{\circ} 28' 40''$	.	9.7943612
sin. dec.	7 7 5		9.0931228	sin.	14 32 49	.	9.3999712

The declination is therefore $7^{\circ} 7' 5''$ south, which (Naut. Alm.) corresponds to March 1st and to October 12.

Also the hour angle SPS' is $29^{\circ} 5' 38''$, which, in time, is $1^h 56^m 22\frac{1}{2}s$, the duration sought.

To find the times of beginning and ending of the twilight, we have, from the equation (3),

cos.	9°	.	9.9946199
cos.	$7^{\circ} 7' 5''$		9.9966399
sin.	95 31 19	.	9.9979800.

The angle nPS' , thus determined, is obtuse, because its opposite side is greater than PS , and this is opposite to a right-angle. This angle, converted into time, is $6^h 22^m 5\frac{1}{4}s$. Adding therefore, to this the angle ZPn , in time, that is, half the duration, or $58^m 11\frac{1}{4}s$, we have $7^h 20^m 16\frac{1}{2}s$, the time when the evening twilight ends. Also, by subtracting the same quantity, we have $5^h 23^m 54s$ for the time when the evening twilight commences. These results respectively taken from 12^h leave the times when the morning twilight begins and ends.

CHAPTER III.

ON THE PRINCIPLES OF NAUTICAL ASTRONOMY.

(73.) IN our chapter on Navigation we have laid down several methods of determining the place of a ship at sea, by help of the account kept on board of its progress through the water, that is, of the course and distance sailed ; and, if confidence could be placed in this account, even when kept with the utmost care, the art of Navigation would be perfect. Such perfection, however, it is hopeless to expect; for it does not seem possible to measure, with strict accuracy, either a ship's rate or the direction in which she moves, both of which may indeed be continually varying. In order, therefore, to determine the place of a ship at sea, with that accuracy which the safety of Navigation requires, it is absolutely necessary that we be furnished with methods entirely independent of the *dead reckoning*, and these methods it is the business of Nautical Astronomy to teach.

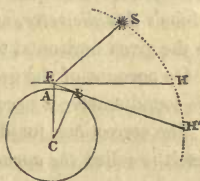
"It must not, however, be understood that the dead reckoning is without its value; on the contrary, when combined with astronomical observations, it is of considerable utility in detecting the existence and velocity of currents, and is indispensably necessary to fill up the short intervals which may occur in unfavorable weather between celestial observations. But the too general practice of relying exclusively upon it cannot be sufficiently deprecated, and numerous instances might be adduced of the fatal consequences of this reliance, in the loss of vessels, from errors of such magnitude that they might have been detected by the most superficial knowledge of nautical astronomy, and the aid of even a good common watch." (*Capt. Kater's Nautical Astronomy in the Ency. Met.*)

*On the Corrections to be applied to the observed Altitudes of
Celestial Objects.*

(74.) The *true altitude* of a celestial object is always understood to mean its angular distance from the rational horizon of the observer. This is not obtained directly by observation; but is the result of certain corrections applied to the observed altitude. These we shall now enumerate and explain.

Of the Dip or the Depression of the Horizon.

(75.) Let E represent the place of the observer's eye, and S the situation of any celestial body; the first object is to obtain its apparent altitude above the horizontal line EH; that is, the angular distance SEH. Now, as to the observer, the *visible horizon* is EBH', the altitude given by the instrument is the angle SEH'; hence we must subtract from this ob-



served altitude the angle HEH', called the *Dip* or *Depression of the horizon*, in order to obtain the apparent altitude SEH.

The angle HEH', or its equal C, is calculated for various elevations, AE of the eye above the surface of the sea from the proportion,

$$CE : EB = \sqrt{EC^2 - CB^2} : : \text{rad.} : \sin. C ;$$

and the results are registered in a table.

Of the Semidiameter.

(76.) When the foregoing correction for dip has been applied, the result will be the apparent altitude of the point observed above the horizontal plane through the observer's eye. If this point be the uppermost or lowermost point of the disc of the sun or moon, a further correction will be necessary to obtain the apparent altitude of the centre; that is, we must apply the angular distance due to the semidiameter. This

quantity, both for the sun and moon, is given in the Nautical Almanack. But in the case of the moon the semidiameter itself requires a small correction depending upon the observed altitude. For the semidiameter, furnished by the Nautical Almanack, is the apparent *horizontal* semidiameter, or the angle it subtends when in the horizon; but as the moon approaches the zenith, her distance from the observer diminishes, and therefore her semidiameter is viewed under a greater angle. As she is nearer to the observer when in the zenith than when in the horizon, by one semidiameter of the earth, and as her distance from the earth's centre is about 60 semidiameters of the earth, the horizontal semidiameter will in the zenith become increased by about $\frac{1}{60}$ th part, and at intermediate elevations the increase will be as the sine of the altitude. On this principle is formed the Table at the end, entitled *Augmentation of the Moon's Semidiameter*, and containing the proper correction to be added to the given horizontal semidiameter to obtain the true semidiameter.

On account of the great distance of the sun, no such correction of his semidiameter is necessary.

The corrections for dip and semidiameter being thus applied, the result is called *the apparent altitude of the centre*. In the case of the stars the only correction for the apparent altitude is the dip. It must, however, be here remarked, that if the centre of the object were visible, and its altitude, instead of that of the limb, were to be taken, we should not, after applying the correction for dip, obtain precisely the same result as that which we have just called the apparent altitude of the centre, but should get a value somewhat less. The reason of this is, that every vertical arc in the heavens is shortened by *refraction*, as we shall shortly explain, so that the centre would not exceed the observed altitude of the lower limb, or fall short of that of the upper, by so great a quantity as the *true* semidiameter. Hence, from the apparent altitude of the centre, as found from applying the *true* semidiameter to the apparent altitude of the limb, a small quantity should in strictness be subtracted, and this small correction becomes necessary when the longitude is to be determined with accuracy. This correction was first proposed by Dr. Thomas Young. A table for it is given at the end.

To obtain the true altitude requires two other corrections, viz. for *refraction* and for *parallax*. The former of these has indeed an effect upon the two preceding corrections, dip, and semidiameter, which

require certain modifications in consequence. One of these we have adverted to above, and the other will be noticed more particularly in the following article.

Of Refraction.

(77.) It is a universal fact in optics, that if a ray of light pass obliquely out of one medium into another of greater density, it will be bent out of its original direction at the point where it enters the new medium, and proceed through it in a direction more nearly perpendicular to its surface at that point. Hence the rays of light, proceeding from the celestial bodies, become bent downwards as soon as they enter the atmosphere, their course being directed more nearly towards the centre of the earth, so that the rays which enter the eye of an observer, and by which any celestial object becomes visible to him, would, if not thus bent down, pass over his head; the object is therefore seen by him above its true place; the angle between this apparent direction, and the true direction of the object, measures the refraction; and, like the correction for dip, it is always *subtractive*; it increases from the horizon, where it is greatest, to the zenith, where it vanishes, as the rays from objects in the zenith enter the atmosphere perpendicularly.

It is the refraction which causes the sun and moon, when near the horizon, to present sometimes an elliptical appearance, the vertical diameter (and, indeed, every oblique diameter) seeming to be shorter than the horizontal, because the lower *limb*, or edge, being more elevated by refraction than the upper, the two are brought, in appearance, more nearly together.

At the end of the volume we have given a table of refractions for different altitudes, from the horizon to the zenith, and adapted to the mean state of the atmosphere; but, as the actual state of the atmosphere generally differs from this, it becomes necessary, where the true altitude of the body is required with the utmost accuracy, to apply a correction to the numbers in this table, so as to adapt them to the existing temperature and density of the atmosphere at the time of observation, as indicated by the thermometer and barometer. This table of corrections is annexed to the table of mean refractions. It should, however, be observed that below 4° the refraction is very variable and uncertain,

and such low altitudes should be avoided as much as possible at sea.

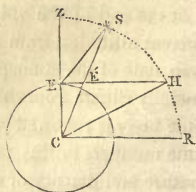
It will be unnecessary to use this annexed table for correcting the altitude of a celestial object when the latitude of the ship is the only object of the observation, as such a correction could seldom make a difference so great as half a mile in the resulting latitude; but, in determining the longitude by the *Lunar Observations*, the neglect of these small corrections would sometimes introduce an error in the resulting longitude of more than 30 miles.

It should be remarked here, that the dip, as determined in article (75), is on the supposition that refraction has not elevated the apparent horizon, but as such is not the case, the dip requires a correction; the amount of this correction is very uncertain, on account of the irregularity of the horizontal refractions, although it is unquestionable that some correction is requisite. It is usual to allow about $\frac{1}{8}$ or $\frac{1}{10}$ of the computed dip for the correction. In our table $\frac{1}{10}$ is allowed, which is according to Dr. Maskelyne, but Lambert and Legendre make it $\frac{1}{14}$.

When the foregoing corrections have been applied to the observed altitude, the result will be the true altitude of the centre above the visible horizon, and it remains now to apply the correction necessary to reduce this to the true altitude of the centre above the rational horizon; that is, to the altitude which the body would have if the observer were situated at the centre of the earth instead of on its surface.

Of the Parallax.

In order to explain the nature and effect of parallax, let S represent the place of the object observed from the surface of the earth, at E; then the angle SEH, that is, the observed angle when corrected for dip semidiameter, and refraction, will be the true altitude of the object, in reference to the observer's sensible horizon EH; and the angle SCR will be the true altitude, in reference to the rational horizon CR; and the difference of these angles is the parallax in altitude. If the body be at H, in the sen-



sible horizon, then the difference of which we speak is the entire angle HCR; this is called the horizontal parallax.

Since the angle SE'H is equal to the angle SCR, we have for the parallax in alt., $SE'H - SEH = ESC$; that is, the parallax is the angle which the semidiameter of the earth subtends at the object; it is obviously greatest in the horizon, and nothing in the zenith, and is the quantity which must be *added* to the true altitude above the sensible horizon, to obtain the true altitude above the rational horizon.

The sun's parallax in altitude is given in a Table at the end; and the moon's horizontal parallax is given for the noon and midnight at Greenwich, of every day of the year, in the Nautical Almanack: and from the horizontal parallax thus obtained the parallax in altitude must be calculated. This is easy; for since in the triangle SEC, we have the proportion

$$SC : EC :: \sin. SEC = \sin. SEZ = \cos. SEH : \sin. ESC ;$$

it follows that the sine of the parallax in altitude varies as the cosine of the altitude, so that, as rad. is to the cosine of the altitude, so is the sine of the horizontal parallax, to the sine of the parallax in altitude. In other words, the log. sine of the horizontal parallax, added to the log. cosine of the altitude, abating 10 from the index, will give the log. sine of the parallax in altitude; but as the parallax is always a very small angle it is usual to substitute the arc for its sine, so that

$$\log. \text{hor. par. in seconds} + \log. \cos. \text{alt.} - 10 = \log. \text{par. in alt. in seconds.}$$

We must observe here that the horizontal parallax, given in the Nautical Almanack, is calculated to the equatorial radius of the earth; and, therefore, except at the equator, a small subtractive correction will be necessary, on account of the spheroidal figure of the earth. A table of such corrections is given at the end, and explained in the introduction to the tables.

Such are the corrections necessary to be applied to the observed altitudes of celestial objects in order to obtain their true altitudes. A few other preliminary, but very simple, and obvious operations must also be performed upon the several quantities taken out of the Nautical Almanack, in order to reduce them to their proper value at the time and place of observation; for the elements furnished by the Nautical Alma-

nack are computed for certain stated epochs, and their values for any intermediate epoch must be found by proportion. But ample directions for these preparatory operations are contained in the "Explanation of the Articles in the Nautical Almanack," by the late Dr. Maskelyne, which accompanies every edition of that work.

Examples of the Corrections.

1. On the 14th of January, 1833, suppose the observed altitude of the sun's lower limb to be $16^{\circ} 24'$, the observer's eye to be 18 feet above the level of the sea, the barometer to stand at 29 inches, and the thermometer at 58° : required the true altitude of the sun's centre.

Observed atl. $\odot$'s L. L.	.	.	16° 36' 4"
Depression of the horizon	.		— 4 4
App. alt. of L. L.	.	.	16 32 0
Refraction	.	.	— 3 14
Correction for Barometer	.	.	— 6.5
Correction for Thermometer	.	.	— 3.2
True alt. of L. L. above visible horizon			16 28 36.3
Sun's semi-diameter (Naut. Alm.)	.		+ 16 17.3
Parallax in altitude	.	.	+ 8.4
True altitude of Sun's centre	.		16 45 2.

2. On the 20th of May, 1833, suppose that in longitude about $77^{\circ} 30'$ west, and lat. about 48° north, at 3^h apparent time, the altitude of the moon's lower limb is observed to be $18^{\circ} 8' 34''$, the height of the eye being 20 feet, the barometer 28.5 inches, and the thermometer 46° : required the true altitude of the sun's centre.

Here the object being the moon, it will be necessary to compute the parallax in altitude, from having the horizontal parallax corresponding to the time at Greenwich.

The horizontal parallax is given in the Nautical Almanack for every noon and midnight; and, therefore, to find it for any other intermediate

time, we must say as 12^h is to its variation in 12^h , so is the proposed time to the variation due to that time.

In like manner must the moon's semidiameter be reduced, by proportion to the time of observation, since it sensibly varies in the course of a few hours. We shall begin, therefore, with finding in this way the true horizontal parallax and semidiameter for the time of observation, reduced to the meridian of Greenwich.

Longitude of the ship in time $5^h 10^m$ after Greenwich time.

Apparent time at ship 3 0

Apparent time at Greenwich 8 10

Hor. par. at noon (Naut. Alm.)	<u>58' 17"</u>	Semidiam. at noon	<u>15' 53"</u>
Hor. par. at midnight	<u>58 31</u>	Semidiam. at midnight	<u>15 57</u>
Variation in 12^h	<u>0 14</u>	Variation in 12^h	<u>0 4</u>
$\therefore 12^h : 8^h 10^m :: 4'' :$	<u>2.7"</u>	$\therefore 12^h : 8^h 10^m :: 14'' :$	<u>9.5"</u>
Semidiameter at noon	<u>15' 53</u>	Hor. par. at noon	<u>58' 17</u>
Hor. semidia. at reduced time	<u>15 55.7"</u>	Hor. par. at reduced time	<u>58 26.5</u>
Augmentation for 18° alt.	<u>5.2</u>		<u>60</u>
True semidiameter	<u>16 0.9</u>	Ditto in seconds	<u>3506.5</u>
		Dim. of par. for lat. 48° —	<u>6.3</u>
		True hor. parallax	<u>3500.2</u>

For the Apparent Altitude.

Observed altitude of Δ 's L. L.	<u>18° 8' 34"</u>
Depression	<u>— 4 17</u>
Semidiameter <i>minus</i> contraction	<u>15 57.9</u>
Apparent alt. Δ 's centre	<u>18 20 14.9.</u>

For the Parallax in Altitude.

cos. $\mathcal{D}$'s app. alt.	$18^{\circ} 20' 15''$	.	9.9773668
hor. parallax	$3500.2''$ log.	.	3.5440929
Par. in altitude	$3322.5''$	.	<u>3.5214597.</u>

For the true Altitude.

Apparent alt. of $\mathcal{D}$'s centre	.	$18^{\circ} 20' 14.9''$
Refraction	.	— 2 54.2
Barometer	.	— 8.8
Thermometer	.	+ 1.4
True alt. above sensible horizon		<u>18 17 13.3</u>
Parallax in altitude	$3322.5'' =$	<u>+ 55 22.5</u>
True alt. of $\mathcal{D}$'s centre	.	<u>19 12 35.8.</u>

These two examples will serve for specimens of the corrections to be applied to an observed altitude, in order to deduce from it the true altitude of the body's centre. In the case of the moon, the corrections, when the utmost accuracy is sought, are rather numerous, as the last example shows. But, in finding the latitude at sea, it is usual to dispense with some of these, more especially with the corrections for temperature, for the contraction of the moon's semidiameter, and for the spheroidal figure of the earth; because an error of a few seconds in the true altitude will introduce no error worth notice in the resulting latitude. When, however, the object of the observers is to deduce the longitude of the ship, all the data, furnished by observation, should be as accurate as possible; for the problem is one of such delicacy that by neglecting to allow for the influence of temperature would alone introduce in some cases an error of from 30 to 40 miles in the longitude.

When the object observed is a star, several of the foregoing correc-

tions vanish: the only corrections, in this case, requisite are those for dip and refraction, modified as usual for the temperature.

(78.) *To determine the latitude at sea from the meridian altitude of any celestial object whose declination is known.*

The determination of the latitude, by a meridian altitude, is the easiest, and in general the safest, method of finding the ship's place on the meridian; for both the observations and the subsequent calculations being few, they are readily performed, and with but little liability to error in the result; this method, therefore, is always to be used at sea, unless foggy or cloudy weather render it impracticable.

The declination of the object observed is supposed to be given in the Nautical Almanack for the meridian of Greenwich; it may therefore be reduced to the meridian of the ship, by means of the longitude by account, which will always be sufficiently accurate for this purpose, although it should differ very considerably from the true longitude, because declination changes so slowly that even an error of an hour in the longitude would cause an error in the declination too small to deserve notice.

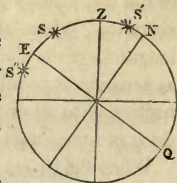
Having then thus found the distance of the object from the equinoctial, and having, by means of the observed altitude properly corrected, obtained the distance of the same object from the ship's zenith, the distance of the zenith from the equinoctial, that is, the latitude, immediately becomes known.

1. Let S be the object observed, the zenith Z being to the north of it, and the object itself north of the equinoctial EQ , then the latitude EZ is equal to the zenith distance, or coaltitude ZS + the declination, and it is north.

2. Let S' be the object, still north of the equinoctial, but so posited that the zenith is south of it, then the latitude EZ is equal to the difference between the zenith distance $S'Z$, and declination $S'E$, and is still north.

3. Let now the object be at S'' , south of the equinoctial, and the zenith to the north of the object, then the latitude EZ is equal to the difference between the zenith distance $S''Z$ and declination $S''E$, and it is north.

We have here assumed the north to be the ele-



vated pole, but if the south be the elevated pole, then we must write south for north, and north for south. Hence the following rule for all cases.

Call the zenith distance north or south, according as the zenith is north or south of the object.

If the zenith distance and declination be of the same name, that is, both north or both south, their sum will be the latitude; but, if of different names, their difference will be the latitude, of the same name as the greater.

EXAMPLES.

1. If on the 2d of May, 1833, the meridian altitude of the sun's lower limb be $47^{\circ} 18'$, height of the eye 20 feet, and longitude by account 32° E.: required the latitude, the sun being south at the time of observation.

Observed alt. of $\odot$'s L. L.	.	$47^{\circ} 18' 0''$
Dip. of the horizon	.	$— 4 17$
App. alt. of $\odot$'s L. L.	.	$46 13 43$

The longitude in time is $2^h 8^m$ east, so that time at Greenwich is $2^h 8^m$ before the noon of the 2d of May; hence, to find the corresponding declination, we have, by the Nautical Almanack,

$$24^h : 2^h 8^m :: 18' 1'' : 1' 38'';$$

so that, $1' 38''$, the variation in $2^h 8^m$, must be subtracted from $15^{\circ} 23' 21''$ N., the declination of the sun on May 2, at noon; hence the proper declination is $15^{\circ} 21' 43''$ N.

Observed alt. of ☉'s L. L.	47° 18' 0"
Dip.	— 4 17
App. alt. of ☉'s L. L.	46 13 43
Refraction	— 56
Parallax	+ 6
Semidiam. (Naut. Alm.)	15 53
True alt of ☉'s centre	46 28 46
Zenith distance	43 31 14 N.
☉'s declination	15 21 43 N.
Latitude	58 52 57 N.

2. On the first of January, 1820, the meridian altitude of Capella was $27^{\circ} 35'$, the zenith being south of the star, and the height of the eye 22 feet; required the latitude.

Observed altitude	27° 35' 0"
Dip.	— 4 30
Apparent altitude	27 30 30
Refraction	— 1 51
True altitude	27 28 39
Zenith distance	62 31 21 S.
Star's dec. (Naut. Alm.)	45 48 39 N.
Latitude	16 42 42 S.

3. On the 19th of February, 1823, the ship being in longitude 40° W., the observed meridian altitude of the moon's lower limb was $55^{\circ} 6'$; the zenith north of the moon; and the height of the eye 16 feet: required the latitude.

Here the time of observation at the ship is not given, it must therefore be calculated, and we have these data for this purpose, viz. that the ship is 40° W. of Greenwich, and that the moon is on its meridian. The following process therefore immediately suggests itself.

The moon passed the merid. of Greenwich Feb. 19 (Naut. Alm.) $6^h 56^m 0^s$

. Feb. 20 $7^h 59^m 0^s$

Interval between the two passages $24^h 1^m 3^s 0^s$

Hence $1^h 3^m$ is the moon's retardation in $25^h 3^m$, and, by proportion using for the longitude 40° W., its value in time $2^h 40^m$, we have,

$$25^h 3^m : 1^h 3^m :: 2^h 40^m : 0^h 6^m 42^s;$$

that is, the moon is retarded $6^m 42^s$ in passing from the meridian of Greenwich to that of the ship, and, therefore, instead of the apparent time at the ship being $6^h 56^m$, as it necessarily would be if there were no retardation, it will be $6^m 42^s$ later. Hence

Apparent time at the ship . . . $7^h 2^m 42^s$

Ship's longitude W. $2^h 40^m 0^s$

Time at Greenwich $9^h 42^m 42^s$

Having thus got the apparent time at Greenwich when the observation was made, we may, by a reference to the Nautical Almanack and a subsequent proportion, find the moon's declination at that time: thus

Moon's declination at Greenwich, Feb. 19 at noon . . . $26^\circ 38' 17''$

. Feb. 19 at midn. $26^\circ 54' 39''$

Change of declination in 12 hours $16' 22''$

$$\therefore 12^h : 9^h 42^m 42^s :: 16' 22'' : 13' 15'';$$

hence $13' 15''$ is the amount of the change of declination, from noon to $9^h 43^m$, on the supposition, however, that the motion of the moon in declination may be considered as equable during the twelve hours. But on account of the irregular motion of the moon, this supposition introduces a sensible error, which may however be corrected by means of the table of "Equation of second Differences," given in the Nautical Almanack, and explained in Dr. Maskelyne's accompanying "Explanation." The correct change of declination is thus found to be $14' 16''$. But from the year 1833, the declination of the moon will be given in the Nautical

Almanack to every three hours, and the change for any shorter interval may then be obtained with the requisite accuracy by proportion, as above. Taking in the present case $14' 16''$ for the correct change, we have

Declination for preceding noon	. 26° 38' 17" N.
Increase of Declination	. 14 16
Declin. at the time of observation	. 26 52 33 N.

Before we can find the proper correction for parallax, we must deduce the apparent altitude of the centre,

Observed altitude of $\mathcal{D}$'s L. L.	. 55° 6' 0"
Dip.	. — 3 50
Semidiameter (Naut. Alm.)	. 16 13
Augmentation for 55° of alt.	. 13
Apparent alt. of $\mathcal{D}$'s centre	. 55 18 36 cos. 9.7552161
Hor. par. in seconds at 9 ^h 43 ^m (Naut. Alm.)	3572 log. 3.5529115
Parallax in altitude in seconds	. 2033 log. 3.3081276

therefore the correction for parallax is $33' 53''$.

Having thus reduced all the corrections to the time of observation, we readily obtain the true altitude, and thence the latitude as follows,

Apparent alt. of $\mathcal{D}$'s centre	. 55° ¹ / ₄₃ 36"
Refraction	. — 40
Parallax in altitude	. 33 53
True altitude	. 55 51 49
Zenith distance	. 34 8 11 N.
Declination	. 26 52 33 N.
Latitude	. 61 0 44 N.

SCHOLIUM.

These examples will, no doubt, be found sufficient to put the student in possession of the method of applying the various corrections to the observed meridian altitude of a celestial object, in order to deduce from it the latitude of the ship. But it should be remarked, that in most works on Nautical Astronomy, subsidiary tables are inserted for the purpose of abridging some of the foregoing corrective operations; such tables, therefore, offer very acceptable aid to the practical navigator. The most esteemed works of this kind are Dr. Mackay's "Treatise on the Theory and Practice of finding the Longitude at Sea"; the "Nautical Tables" of J. De Mendoza Rios, and Mr. Riddle's book on Navigation and Nautical Astronomy.

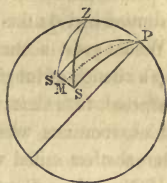
It should also be observed here, that in the preceding examples the celestial object is supposed to be on the meridian *above* the pole; that is, to be higher than the elevated pole. But, if a meridian altitude be taken below the pole, which may be done if the object is *circumpolar*, or so near to the elevated pole as to perform its apparent daily revolution about it without passing below the horizon, then the latitude of the place will be equal to the sum of the true altitude, and the codeclination or polar distance of the object; for this sum will obviously measure the elevation of the pole above the horizon, which is equal to the latitude.

(79.) *To determine the latitude at sea, by means of two altitudes of the sun, and the time between the observations.*

In the preceding article we have shewn how to determine the latitude of the ship by the meridian altitude of the sun, or of any other heavenly body, whose declination may be found. But, as already remarked, the object we wish to observe may be obscured when it comes to the meridian, and this may happen for many days together, although it may be frequently visible at other times of the day. As therefore the opportunity for a meridian observation cannot be depended upon, it becomes an important problem to determine the latitude at sea, by observations made out of the meridian; and considerable attention has accordingly been paid, by scientific persons, to the method of finding the latitude by "double altitudes," and various tables have been computed to facilitate

the operation. But the direct method, by spherical trigonometry, though rather long, involving three spherical triangles, will be more readily remembered, and more easily applied by persons familiar with the rules and formulas of Trigonometry, than any indirect or approximative process; we shall therefore explain the direct method.

Let P be the elevated pole, Z the zenith of the ship, and S, S' , the two places of the sun, when the altitudes are taken. Then, drawing the great circle arcs as in the figure, we shall have these given quantities, viz. the codeclinations PS, PS' ; the coaltitudes ZS, ZS' , and the hour angle SPS' , which measures the interval between the observations; and the quantify sought is the colatitude ZP . Now, in the triangle PSS' , we have given two sides and the included angle to find the third side SS' , and one of the remaining angles, say the angle PSS' . In the triangle ZSS' we have given the three sides to find the angle $S'SZ$; having then the angles $PSS', S'SZ$, the angle ZSP becomes known, so that we have, lastly, two sides and the included angle in the triangle ZP , to find the third side ZP .



Before the application of the trigonometrical process, the observed altitudes must, of course, be reduced to the true altitudes, as in the preceding examples. Moreover, as the ship most probably sails during the interval of the observation, an additional reduction becomes necessary; the first altitude must be reduced to what it would have been if taken at the place where the second was taken: this correction will be known if we know the number of minutes the ship has sailed directly towards or directly from the sun, upon leaving the place where the first observation was made. To find this, take the angle included between the ship's course and the sun's bearing, at the first observation; and considering this angle as a course, and the distance sailed as the corresponding distance, find by the traverse table, or by the operation of plane sailing, the difference of latitude, which will be the amount of the approach to, or departure from, the sun. This must be added to the first altitude if the angle is less than 90° , because the ship will have approached towards the sun; but it must be subtracted when the angle exceeds 90° . If the angle is 90° no correction for the ship's change of place will be necessary.

The truth of this correction will be immediately seen by considering that if the sun's centre were the elevated pole, what is in reality the colatitude would then be the colatitude, and, therefore, that, by whatever quantity this latter is increased or diminished by the ship's motion, on the one hypothesis, by the same quantity will the former be increased or diminished on the other hypothesis.

Where great accuracy is aimed at, account should be taken of the ship's change of longitude during the interval of the observations; when converted into time it must be added to the interval of time between the observations when the ship has sailed eastward, and subtracted when she has sailed westward. This correction is very easily applied.

Having thus mentioned the necessary preparative corrections, we shall now give an example of the trigonometrical operation.

EXAMPLES.

Let the two zenith distances corrected be (see last fig.) $ZS = 73^\circ 54' 13''$, $ZS' = 47^\circ 42' 51''$, the corresponding declinations $8^\circ 18'$ and $8^\circ 15'$ north, and the interval of time three hours; to determine the latitude.

Considering SS' to be the base of an isosceles spherical triangle, of which one of the equal sides is $\frac{1}{2}(PS + PS') = 81^\circ 43' 30''$, and the vertical angle equal to 3^h or 45° , let the perpendicular PM be drawn, then we have in the triangle PMS right angled at M , $PS = 81^\circ 43' 30''$, and $P = \frac{45^\circ}{2} = 22^\circ 30'$; given to find $SM = \frac{1}{2} SS'$ as follows.

1. *To find SS' from the triangle PMS .*

sin. PS	$81^\circ 43' 30''$	.	.	9.9954547
sin. P	$22\ 30\ 0$	.	.	9.5828397
sin. SM	$22\ 15\ 11.3$	.	.	9.5782944
				2
$SS' =$	$44\ 30\ 22.6$			

II. To find PSS' from the triangle PSS'.

sin. SS' 44° 30' 22.6"	arith. comp. 0.1542898
sin. PS' 81 45 0	9.9954822
sin. SPS' 45 0 0	9.8494850
sin. PSS' 86 38 58	9.9992570.

This angle is acute like its opposite side, (see art. 60.)

III. To find ZSS' in the triangle ZSS'.

$$ZS' 47^{\circ} 45' 51''$$

$$\sin. ZS 73 54 13 \quad \text{arith. comp. } 0.0173686$$

$$\sin. SS' 44 30 22.6 \quad \text{arith. comp. } 0.1542898$$

$$2) 166 10 26.6$$

$$\frac{1}{2} \text{ Sum} = 83 5 13.3$$

$$\sin. (\frac{1}{2} \text{ Sum} - ZS) 9 11 0.3 \quad 9.2030206$$

$$\sin. (\frac{1}{2} \text{ Sum} - SS') 38 34 50.7 \quad 9.7949179$$

$$2) 19.1695969$$

$$\sin. \frac{1}{2} ZSS' 22 36 26.4 \quad 9.5847985$$

$$\therefore ZSS' = 45^{\circ} 12' 52.8''$$

$$PSS' = 86 38 58$$

$$PSZ = 41 26 5.2.$$

iv. To find ZP in the triangle ZSP.

tan. PS $81^{\circ}42' 0''$	.	10.8359917,	cos. PS	.	9.1594354
cos. PSZ $41\ 26\ 5.2$	.	9.8748930,	sin. ω , ar. comp.	.	7.189551
cot. ω $11\ 0\ 41.2$	.	10.7108847,	sin. ($\omega + ZS$)	.	9.9982874
73 54 54.2			sin. $48^{\circ}49'59.7''$	.	9.8766779
$\omega + ZS = 84\ 54\ 54.2$					

Hence the latitude is $48^{\circ}50'$.

2. The two corrected altitudes are $42^{\circ}14'$ and $16^{\circ}5'47''$, the corresponding declinations $8^{\circ}16'30''$ and $8^{\circ}15'$, and the time between the observations 3 hours; required the latitude of the place.

The latitude is $48^{\circ}54'27''$ N.

Upon the same principles may the latitude be determined from the altitudes of two fixed stars, taken at the same time; in this case S, S', in the preceding figure, will represent the two stars; PS, PS', their known polar distances, and the angle SPS', the difference of their right ascensions; the same quantities are therefore given as in the case of the sun, but, as in the case of two stars PS, PS', may differ very considerably; SS' cannot be considered as the base of an isosceles triangle, but must be computed from the other two sides and their included angle. In the Nautical Almanack for 1825 Dr. Brinkley has computed for 1822, and tabulated, the distances SS' for certain pairs of stars, conveniently situated for observation, and has annexed the change of distance corresponding to 10 years. The same table shows also the difference of right ascension for each pair of stars, with the change in 10 years; so that by help of this table the computation for finding the latitude from the simultaneous altitudes of two fixed stars becomes considerably abridged.

For other methods of determining the latitude, the student may consult "Mackay on the Longitude," vol. I., and Captain Kater's Nautical Astronomy, in the Ency. Metropolitana, &c.

On finding the Longitude by the Lunar Observations.

(80.) There are several astronomical methods of determining the longitude of a place which cannot be accurately employed at sea, on account of the great difficulty of managing a telescope on shipboard; we shall not, therefore, enter here into any explanation of these methods, but shall confine ourselves to the *lunar method* of determining the longitude, which is justly regarded as the principal problem in Nautical Astronomy. Before entering upon the solution of this problem it will be necessary to make a few introductory remarks.

The determination of the longitude of a place always requires the solution of these two problems, viz. 1st, to determine the time at the place at any instant; and, 2d, to determine the time at the first meridian, at the same instant; for the difference of the times converted into degrees, at the rate of 15° to an hour, will obviously give the longitude.

When the latitude of the place is known, (and it may be found by the methods already explained,) the time may be computed from the altitude of any celestial object whose declination is known; for the co-altitude, codeclination, and colatitude, will be three sides of a spherical triangle given to find the hour angle, comprised between the codeclination and the colatitude. But to find the time at Greenwich requires the aid of additional data, besides those furnished by observations made at the place. The Greenwich time may, indeed, be obtained at once, independently of any observations at the place, by means of a chronometer, carefully regulated to Greenwich time, provided it be subject to no irregularities after having been once properly adjusted. A ship furnished with such a timepiece always carries the Greenwich time with her,*

* As chronometers show mean time, the equation of time must be applied to obtain the *apparent* time at Greenwich.

and the longitude then becomes reduced to the problem of finding the time at the place. Chronometers are now brought to such a state of perfection that very great dependence can be placed on them, and they are accordingly always taken out on long voyages for the purpose of showing the Greenwich time, and are thus of great use to the mariner. Still, however, as the most perfect contrivance of human art is subject to accident, and the more delicate the machine the more liable is it to disarrangement, from causes which we may not be able to control, it becomes highly desirable, in so important a matter as finding the place of a ship at sea, to be possessed of methods altogether beyond the influence of terrestrial vicissitudes, and such methods the celestial motions alone can supply.

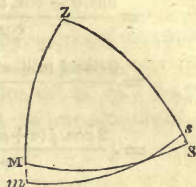
The angular motion of the moon in her orbit is more rapid than that of any other celestial body, and sufficiently great to render the portion of its path passed over in so short a time as two or three seconds, a measurable quantity even with a small portable instrument (the sextant).

It is obvious, therefore, that if the distance of the moon's centre from any celestial body, in or near her path, be computed for any Greenwich time, and this distance be found the same as that given by actual observation at any place, then the difference between the time of observing the phenomenon and the time at Greenwich, when it was predicted to happen, will give the longitude of the place of observation. Now in the Nautical Almanack the distances of the moon from the sun, and from several of the fixed stars near her path, are given for every three hours of apparent Greenwich time, and for several years to come; and the Greenwich time, corresponding to any intermediate distance, is obtainable by simple proportion with all requisite accuracy; so that by means of the Nautical Almanack we may always determine the time at Greenwich when any distance observed at sea was taken.

The distances inserted in the Nautical Almanack are the true angular distances between the centres of the bodies, the observer being considered as at the centre of the earth, and to the true distance therefore every observed distance must be reduced; it is this reduction which constitutes the trigonometrical difficulties of the problem; and it consists in

Clearing the lunar distance from the effects of parallax and refraction; how to do this it is now our business to explain.

Let m, s , be the observed places of the moon and sun, or of the moon and a fixed star, and let M, S , be their true places. M will be above m , because the moon is depressed by parallax more than it is elevated by refraction; but S will be below s , because the sun is more elevated by refraction than it is depressed by parallax. Observation gives the apparent distance ms , and the apparent zenith distances Zm, Zs : by applying the proper corrections to these latter we also deduce the true zenith distances ZM, ZS , and with these data we are to determine the true distance, MS , by computation.



Put d for the apparent distance.

D true distance.

a, a' apparent altitudes.

A, A' true altitudes.

Then in the triangle MZS , we have,

$$\cos. Z = \frac{\cos. D - \sin. A \sin. A'}{\cos. A \cos. A'};$$

and in the triangle mZs ,

$$\cos. Z = \frac{\cos. d - \sin. a \sin. a'}{\cos. a \cos. a'};$$

hence, for the determination of D , we have this equation, viz.

$$\frac{\cos. D - \sin. A \sin. A'}{\cos. A \cos. A'} = \frac{\cos. d - \sin. a \sin. a'}{\cos. a \cos. a'};$$

from which we immediately get

$$\begin{aligned}
\cos. D &= (\cos. d - \sin. a \sin. a') \frac{\cos. A \cos. A'}{\cos. a \cos. a'} + \sin. A \sin. A' \\
&= \frac{\cos. d + \cos. (a + a') - \cos. a \cos. a'}{\cos. a \cos. a'} \cos. A \cos. A' + \sin. A \sin. A' \\
&= \frac{2 \cos. \frac{1}{2} (a + a' + d) \cos. \frac{1}{2} (\overline{a + a'} \sim d) \cos. A \cos. A'}{\cos. a \cos. a'} - \cos. (A + A') \quad (1) \\
&= \left\{ \frac{2 \cos. \frac{1}{2} (a + a' + d) \cos. \frac{1}{2} (\overline{a + a'} \sim d) \cos. A \cos. A'}{\cos. a \cos. a' \cos. (A + A')} - 1 \right\} \cos. (A + A');
\end{aligned}$$

or calling the first term within the brackets $2 \cos.^2 F$,

$$\cos. D = (2 \cos.^2 F - 1) \cos. (A + A') = \cos. 2 F \cos. (A + A') \dots (2).$$

The formulas marked (1) and (2) are both of them convenient for the computation of D ; a third formula may be obtained from (1), as follows. Subtract each side of (1) from 1; then since (p. 55),

$$1 - \cos. D = 2 \sin.^2 \frac{1}{2} D, \quad 1 + \cos. (A + A') = 2 \cos.^2 \frac{1}{2} (A + A'),$$

we have, after dividing by 2,

$$\begin{aligned}
\sin.^2 \frac{1}{2} D &= \cos.^2 \frac{1}{2} (A + A') - \frac{\cos. \frac{1}{2} (a + a' + d) \cos. \frac{1}{2} (\overline{a + a'} \sim d) \cos. A \cos. A'}{\cos. a \cos. a'} \\
&= \cos.^2 \frac{1}{2} (A + A') \left\{ 1 - \frac{\cos. \frac{1}{2} (a + a' + d) \cos. \frac{1}{2} (\overline{a + a'} \sim d) \cos. A \cos. A'}{\cos. a \cos. a' \cos.^2 \frac{1}{2} (A + A')} \right\};
\end{aligned}$$

or, calling the second term within the brackets $\sin.^2 \theta$,

$$\sin.^2 \frac{1}{2} D = \cos.^2 \frac{1}{2} (A + A') \cos.^2 \theta$$

$$\therefore \sin. \frac{1}{2} D = \cos. \frac{1}{2} (A + A') \cos. \theta \dots (3).$$

This latter is *Borda's* formula.

We shall solve an example by each of these formulas.

EXAMPLES.

1. Suppose the apparent distance between the centres of the sun and moon to be $83^{\circ} 57' 33''$, the apparent altitude of the moon's centre $27^{\circ} 34' 5''$, the apparent altitude of the sun's centre $48^{\circ} 27' 32''$, the true altitude of the moon's centre $28^{\circ} 20' 48''$, and the true altitude of the sun's centre $48^{\circ} 26' 49''$; then we have

$$d = 83^{\circ} 57' 33'', \quad a = 27^{\circ} 34' 5'', \quad a' = 48^{\circ} 27' 32''$$

$$A = 28^{\circ} 20' 48'', \quad A' = 48^{\circ} 26' 49'';$$

and the computation for D , by the first formula, is as follows:

d	$83^{\circ} 57' 33''$		
a	$27 \ 34 \ 5$	.	comp. cos. 0523390
a'	$48 \ 27 \ 32$	.	comp. cos. 1783835
2)	$159 \ 59 \ 10$	.	log. 2 3010300
$\frac{1}{2}$ sum	$79 \ 59 \ 35$	.	cos. 9.2399686
$\frac{1}{2}$ sum $\sim d$	$3 \ 57 \ 58$	.	cos. 9.9989587
A	$28 \ 20 \ 48$	.	cos. 9.9445275
A'	$48 \ 26 \ 49$	.	cos. 9.8217187
(Reject 40 from index) $\bar{1} \cdot 5369260 = \log. \cdot 3442921 +$			
$A + A'$	$76 \ 47 \ 37$	.	nat. cos. $\cdot 2284595 -$
True distance $83^{\circ} 20' 54''$ nat. cos. $\cdot 1158326.$			

By glancing at the formula (1), we see that 30 must be rejected from the sum of the above column of logarithms, so that the logarithmic line resulting from the process is $9 \cdot 5369260$. Now, as in the table of log. sines, log. cosines, &c., the radius is supposed to be 10^{10} , of which the log. is 10, and, in the table of natural sines, cosines, &c., the rad. is 1,

of which the log. is 0; it follows that when we wish to find, by help of a table of the logarithms of numbers, the natural trigonometrical line corresponding to any logarithmic one, we must diminish this latter by 10, and enter the table with the remainder. Hence the sum of the foregoing column of logarithms must be diminished by 40, and the remainder will be truly the logarithm of the natural number represented by the first term in the second member of the equation (1). If this natural number be less than nat. cos. $(A + A')$, which is to be subtracted from it, the remainder will be negative, in which case D will be obtuse.

By the second formula the process is as follows:

$d \ 83^{\circ} 57' 33''$	
$d \ 27 \ 34 \ 5$	comp. cos. 0.0523390
$a' \ 48 \ 27 \ 32$	comp. cos. 0.1783835
2) 159 59 10	
$\frac{1}{2}$ sum 79 59 35	cos. 9.2399686
$\frac{1}{2}$ sum $\sim d \ 3 \ 57 \ 58$	cos. 9.9989587
$A \ 28 \ 20 \ 48$	cos. 9.9445275
$A' \ 48 \ 26 \ 49$	cos. 9.8217187
$A + A' \ 76 \ 47 \ 37$	comp. cos. 0.6411909 —
2) 19.8770869	
$F \ 29 \ 46 \ 3$	cos. 9.9385434
$2F \ 59 \ 32 \ 6$	cos. 9.7050182 +
True distance $83^{\circ} 20' 54''$ cos. 9.0638273.	

In adding up the logarithms to find $\cos. 2F$, 20 must be rejected from the index; and the logarithm marked —, is to be subtracted from that marked +. Moreover, if $A + A'$ and $2F$ are both acute or both obtuse, D will be acute, otherwise it will be obtuse.

We shall now exhibit the process by *Borda's formula*.

d	83° 51' 33"		
a	27 35 5	.	comp. cos. 0.0523390
a'	48 27 32	.	comp. cos. 0.1783835
	2) 159 59 10		
$\frac{1}{2}$ sum	79 59 35	.	cos. 9.2399686.
$\frac{1}{2}$ sum $\sim d$	3 57 58	.	cos. 9.9989587
A	28 20 48	.	cos. 9.9445275
A'	48 26 49	.	cos. 9.8217187
A + A'	76 47 37		2) 39.2358960
			19.6179480
$\frac{1}{2}(A + A')$	38 23 48 $\frac{1}{2}$	.	cos. 9.8941654 +
θ	31 57 53 $\frac{1}{2}$	.	sin. 9.7237826
θ	.	.	cos. 9.9285870 +
$\frac{1}{2} D$	41 40 27	.	sin. 9.8227524

$\therefore D = 83 \ 20 \ 54$, the true distance.

An estimate may now be formed of the relative advantages of these three methods, as regards practical facility. We are inclined to prefer the first method, which we believe is new, as fewer references to the tables are requisite, and as moreover there are no arithmetical operations required, besides those which are actually exhibited. The second and third methods seem to offer nearly equal advantages; in the first of these, however, it may be observed that the trigonometrical lines involved are all of one name, viz. cosines, and that the final reference to the tables gives the true distance instead of its half, as in the last method.

Each of the foregoing processes may be shortened by using a subsidiary table, containing the various values of the expression $\frac{\cos. A \cos. A'}{\cos. a \cos. a'}$. Such a table computed to every degree of the moon's

apparent altitude, and to every 10 seconds of her horizontal parallax, forms Table ix. of the *Requisite Tables*, published by order of the Commissioners of Longitude. But a more complete table of this kind is given in the second volume of Dr. Mackay's work, on the Longitude. If each number in this table were increased by the constant number 3010300, the table itself would become somewhat simplified, and the process of clearing the distance by our first method would be rendered remarkably short and convenient.

The preceding example is taken from Woodhouse's *Astronomy*, part II., p. 859, where the day of observation is stated to be June 5, 1793. Now by the *Nautical Almanack*, for that year, we have

$$\begin{array}{rcl}
 \text{Distance at } 15^h 83^\circ 6' 1'', & \text{Also at time of observation} & D = 83^\circ 20' 55'' \\
 \text{at } 18^h 84 \ 28 \ 26 & & \text{at } 15^h \ D = 83 \ 6 \ 1 \\
 \hline
 \text{Increase in } 3^h \ 1 \ 22 \ 25, & \text{Increase between } 15^h \text{ and time of obs.} & 0 \ 14 \ 54 \\
 \hline
 \therefore 1^\circ 22' 25'' : 14' 54'' :: 3^h : 32^m 33^s.
 \end{array}$$

Hence, when the observation was made, the apparent time at Greenwich was $15^h 32^m 33^s$.

To find the time at the ship, requires that we know the latitude of the place and the sun's declination. The former, therefore, must have been previously ascertained, and the latter may now be found by means of the apparent Greenwich time just deduced, and the *Nautical Almanack*. We shall suppose the latitude to be $10^\circ 16' 40''$ S.; the sun's declination will be $23^\circ 22' 28''$, and taking the true altitude of the sun $= 48^\circ 46' 49''$, we shall thus have, in order to find the time, three sides of a spherical triangle to find an angle. The computation is as follows.

coalt. $41^{\circ}33'11''$

sin. colat. $79\ 43\ 20$ arith. comp. 0.0070251

sin. sun's polar dist. $113\ 22\ 48$ arith. comp. 0.0372078

2) $234\ 39\ 19$

$117\ 19\ 39.5$

sin. $37\ 36\ 19.5$. 9.7854864

sin. $3\ 56\ 51.5$. 8.8378712

2) 18.6675905

sin. $12\ 27\ 17\frac{1}{2}$. 9.3337902

2

Hour angle $= 24\ 54\ 35 = 1^h\ 39^m\ 35.3$ in time.

Time at Greenwich

$15\ 32\ 33$

L. in time, *reckoning westward*.

$13\ 52\ 54.7$.

Or, subtracting this from 24 hours, we have $10^h\ 7^m\ 4.3$, for the longitude *east*, in time, and therefore the longitude in degrees is $151^{\circ}\ 46'\ 4\frac{1}{2}''$ E.

2. Given the apparent altitude of the moon's centre $8^{\circ}\ 26'\ 13''$, the true altitude $9^{\circ}\ 20'\ 45''$, the apparent altitude of a star $35^{\circ}\ 40'$, the true altitude $35^{\circ}\ 38'\ 49''$, and their apparent distance $31^{\circ}\ 13'\ 26''$; to determine the true distance.

The true distance is $30^{\circ}\ 23'\ 56''$.

Those who are desirous of entering more at large into the problem of the Longitude, and of becoming acquainted with the best methods of shortening the computation by the aid of subsidiary tables, may advantageously consult, besides the works already referred to, the Quarto Tables of J. De Mendoza Rios, Lynn's Navigation Tables, Captain Kater's Treatise on Nautical Astronomy, in the Encyclopædia Metropolitana, Kerrigan's Navigator's Guide and Nautical Tables, and Dr. Myers's translation of Rossel on the Longitude.

Variation of the Compass.

(81.) We shall conclude this part of our subject by briefly considering the methods of finding the variation of the compass, or the quantity by which the north point, as shown by the compass, varies easterly or westerly from the true north point of the horizon.

The solution of this problem merely requires that we find by computation, or by some means independent of the compass, the *bearing* of a celestial object, that we observe the bearing by the compass, and then take the difference of the two. The problem resolves itself, therefore, into two cases, the object whose bearing is sought being either in the horizon or above it: in the one case we have to compute its *amplitude*, and in the other its *azimuth*.

The computation of the amplitude is simply determining the hypotenuse of a right-angled triangle, of which one side is given, viz. the declination of the object, as also the angle opposite to it, viz. the colatitude. The computation of the azimuth requires the solution of an oblique spherical triangle, the three sides being given to find an angle; the three given sides are the colatitude; the zenith distance of the object and its polar distance: and the azimuth being measured by the angle at the zenith opposite the polar distance, this is the angle sought. We shall give an example in each of these cases of the problem.

EXAMPLES.

1. In January 1830, at latitude $27^{\circ} 36' N.$, the rising amplitude of Aldebaran was, by the compass* E. $23^{\circ} 30' N.$; required the variation.

By the Nautical Almanack the declination of Aldebaran is $16^{\circ} 9' 37'' N.$, therefore since $\text{Rad.} \times \sin. \text{dec.} = \sin. \text{amp.} \times \cos. \text{lat.}$, the computation is as follows.

* The compass amplitude must be taken when the apparent altitude of the object is equal to the depression of the horizon.

$$\sin. \text{declination } 16^{\circ} 9' 37'' \quad . \quad . \quad 9.4445527$$

$$\cos. \text{latitude } 27 \ 36 \quad . \quad . \quad 9.9475335$$

$$\sin. \text{Amplitude E. } 18 \ 18 \ 17 \ \text{N.} \quad . \quad . \quad 9.4970292$$

$$\text{Magnetic Amplitude E. } 23 \ 30 \ 0 \ \text{N.}$$

$$\text{Variation } 5 \ 11 \ 43.$$

As the object is farther from the magnetic east than from the true east, the magnetic east has therefore advanced towards the south, and therefore the magnetic north towards the east; hence the variation is $5^{\circ} 11' 43''$ E.

2. In latitude $48^{\circ} 50'$ north, the true altitude of the sun's centre was $22^{\circ} 2'$, the declination at the time was $10^{\circ} 12'$ S., and its magnetic bearing $161^{\circ} 32'$ east. Required the variation.

$$\odot\text{'s polar distance } 100^{\circ} 12'$$

$$\sin. \text{zenith distance } 67 \ 58 \text{ arith. comp. } 0.0329363$$

$$\sin. \text{colatitude } 41 \ 10 \text{ arith. comp. } 0.1816080$$

$$2)209 \ 20$$

$$\sin. \frac{1}{2} \text{ S } 104 \ 40 \quad . \quad 9.9856129$$

$$\sin. (\frac{1}{2} \text{ S} - \text{pol. dist.}) \ 4 \ 28 \quad . \quad 8.8914209$$

$$2)19.0915781$$

$$\cos. 69^{\circ} 25' 40'' \quad . \quad 9.5457895$$

$$2$$

$$\odot\text{'s true azimuth N. } 138 \ 51 \ 20 \ \text{E.}$$

$$\text{Observed azimuth N. } 161 \ 32 \ 0 \ \text{E.}$$

$$22 \ 40 \ 40 \ \text{West.}$$

The variation is west, because the sun's observed distance from the

north, measured easterly, being greater than its true distance, intimates that the north point of the compass has approached towards the west.

3. In latitude $48^{\circ} 20'$ north, the star Rigel was observed to set $9^{\circ} 50'$ to the northward of the west point of the compass; required the variation, the declination of Rigel being $8^{\circ} 25' S$.

Variation $22^{\circ} 33'$ West.

4. In latitude $50^{\circ} 12'$ north, when the sun's declination was $11^{\circ} 28' 53' N$., its true altitude was found to be $37^{\circ} 0' 16''$, and the observed azimuth $S. 31^{\circ} E$.; required the variation of the compass.

Variation $28^{\circ} 2'$ West.

END OF PART III.

PART IV.

MISCELLANEOUS TRIGONOMETRICAL INQUIRIES.

(82.) WE now come to the final *part* of our subject, in which we propose to bring together several miscellaneous particulars which properly come under consideration in a treatise on Trigonometry. One or two of these, especially those which relate to certain compendious solutions of plane triangles, and to the trigonometrical lines of small arcs, might have been introduced much earlier, although we have preferred to postpone their consideration for a supplementary chapter, agreeing with *Woodhouse*, that it is better for the student first “to attend solely to the general solutions, and to postpone to a time of leisure and of acquired knowledge the consideration of the methods that are either more expeditious or are adapted to particular exigencies.”

CHAPTER I.

ON THE SOLUTIONS OF CERTAIN CASES OF PLANE TRIANGLES, AND ON DETERMINING THE TRIGONOMETRICAL LINES OF SMALL ARCS.

PROBLEM I.

(83.) Given two sides and the included angle of a plane triangle, to determine the third side, without finding the remaining angles.

The general expression for the side c , in terms of the two sides a, b , and the included angle C , is (17),

$$\begin{aligned}
 c^2 &= a^2 + b^2 - 2ab \cos. C \\
 &= (a - b)^2 + 2ab(1 - \cos. C) \\
 &= (a - b)^2 + 2ab \cdot 2 \sin.^2 \frac{1}{2} C \\
 &= (a - b)^2 \left\{ 1 + \frac{4ab}{(a - b)^2} \sin.^2 \frac{1}{2} C \right\}.
 \end{aligned}$$

Assume the second term within the brackets equal to $\tan.^2 \theta$ then,
 since $1 + \tan.^2 \theta = \sec.^2 \theta = \frac{\text{rad.}^2}{\cos.^2 \theta}$, we have

$$c = (a - b) \frac{\text{rad.}}{\cos. \theta}.$$

Hence c is determined by these two formulas, viz.

$$\log. \tan. \theta = \log. 2 + \frac{1}{2} \log. a + \frac{1}{2} \log. b + \log. \sin. \frac{1}{2} C - \log. (a - b)$$

$$\log. c = \log. (a - b) + 10 - \log. \cos. \theta.$$

EXAMPLE.

Given $a = 562$, $b = 320$, and $C = 128^\circ 4'$, to find c .

log. 2	0.3010300		
$\frac{1}{2} \log. 562$	1.3748681		
$\frac{1}{2} \log. 320$	1.2525750		
log. sin. $64^\circ 2'$	9.9537833		
ar. comp. log. 242	7.6161846,	log. 242 + 10 .	12.3838154
log. tan. θ	10.4984410	$\therefore \log. \cos. \theta =$	9.4807177
log. c	800.01	.	2.9030977.

PROBLEM II.

Given the logarithms of two sides of a plane triangle, as also the included angle, to determine the remaining angles.

Let $\log. a$, $\log. b$, and C , be given. Suppose a greater than b , and assume $r \frac{a}{b} = \tan. \theta$; then $\tan. \theta$ being greater than 1, θ will exceed 45° . Also (19).

$$\begin{aligned} \tan. \frac{1}{2} (A - B) &= \frac{a - b}{a + b} \cot. \frac{1}{2} C = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} \cot. \frac{1}{2} C \\ &= \frac{\tan. \theta - 1}{\tan. \theta + 1} \cot. \frac{1}{2} C = \tan. (\theta - 45^\circ) \cot. \frac{1}{2} C \text{ (p. 48).} \end{aligned}$$

Hence, introducing the radius, $A - B$ is determined by these two formulas, viz.

$$\log. \tan. \theta = 10 + \log. a - \log. b$$

$$\log. \tan. \frac{1}{2} (A - B) = \log. \tan. (\theta - 45^\circ) + \log. \cot. \frac{1}{2} C - 10.$$

Thus, taking the example in last problem, we have

$$\begin{array}{rcl} 10 + \log. 562 & . & 12.7497363 \\ \log. 320 & . & 2.5051500 \\ \hline \log. \tan. \theta & . & 10.2445863 \therefore \theta = 60^\circ 20' 35'' \\ \hline \therefore \theta - 45^\circ & = & 15^\circ 20' 35''. \end{array}$$

Again,

$$\begin{array}{rcl} \log. \tan. 15^\circ 20' 35'' & 9.4383476 & \\ \log. \cot. 64 \quad 2 & . & 9.6875402 \\ \hline \log. \tan. \frac{1}{2} (A - B) & 9.1258878 & \therefore \frac{1}{2} (A - B) = 7^\circ 36' 40'' \\ \hline \frac{1}{2} (A + B) & = & 25 \quad 58 \\ \hline A & = & 33 \quad 34 \quad 40 \\ \hline B & = & 18 \quad 21 \quad 20. \end{array}$$

This method of determining the angles A and B will always be the shortest, when instead of the sides their logarithms are given. Thus the solution of problem x., p. 44, becomes much facilitated by the application of this process.

PROBLEM III.

To determine the area of a plane triangle when any three parts except the three angles are given.

1. Let two sides, b, c , and the included angle A, be given. (See fig. p. 16.)

The area of the triangle is expressed by $\frac{1}{2} AB \cdot CD$; but $CD = AC \sin. A$; hence the expression for the area, in terms of the given quantities, is

$$\text{Area} = \frac{1}{2} bc \sin. A.$$

2. Let two angles, A, B, and the interjacent side c , be given.

Then, since

$$\sin. C : \sin. B :: c : b,$$

we have

$$b = \frac{\sin. B}{\sin. C} c \therefore bc \sin. A = \frac{\sin. A \sin. B}{\sin. C} c^2;$$

hence the expression for the area is

$$\text{Area} = \frac{\sin. A \sin. B}{2 \sin. C} c^2.$$

3. Let the three sides be given.

By art. (20),

$$\sin. \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2} S - b)(\frac{1}{2} S - c)}{bc}}$$

$$\cos. \frac{1}{2} A = \sqrt{\frac{\frac{1}{2} S (\frac{1}{2} S - a)}{bc}}$$

$$\therefore 2 \sin. \frac{1}{2} A \cos. \frac{1}{2} A, \text{ or (art. 31) } \sin. A = \frac{2}{bc} \sqrt{\frac{1}{2} S (\frac{1}{2} S - a)(\frac{1}{2} S - b)(\frac{1}{2} S - c)}.$$

Consequently, by substituting this value of $\sin. A$ in the first ex-

pression, we have

$$\text{Area} = \sqrt{\frac{1}{2}S(\frac{1}{2}S - a)(\frac{1}{2}S - b)(\frac{1}{2}S - c)};$$

which formula furnishes the well known rule, given in all books on mensuration, for the area of a triangle when the three sides are given. (See Geom. p. 202.)

These expressions for the area of a plane triangle are all adapted to logarithmic computation.

PROBLEM IV.

To find the logarithmic sine of a very small arc.

By article (30) the expression for the sine of any arc x is

$$\sin. x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

Now as the length of an arc of one degree is .01745329, (see p. 54,) it is plain that, even when x is so great as this, the third term of the above series can have no significant figure in the first ten places of decimals.

Retaining therefore only the first two terms, we have, when x is small,

$$\sin. x = x - \frac{x^3}{1 \cdot 2 \cdot 3} = x(1 - \frac{x^2}{2 \cdot 3}) = x\{1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4}\}^{\frac{1}{2}} \text{ nearly};$$

that is, (p. 53,)

$$\sin. x = x \cos.^{\frac{1}{2}} x;$$

hence, by introducing the radius,

$$\log. \sin. x = \log. x - \frac{1}{2} (10 - \log. \cos. x) \dots (1).$$

Let the arc x contain n seconds, then

$$x = \frac{n \cdot \pi}{180 \times 60 \times 60};$$

hence, by introducing the radius,

$$\begin{aligned}\log. x &= \log. n + \log. 3.14159, \text{ \&c. } + 10 - \log. \overline{180 \times 60^2} \\ &= \log. n + 4.6855749;\end{aligned}$$

therefore, from (1),

$$\log. \sin. x = \log. n + 4.6855749 - \frac{1}{3} \text{ arith. comp. log. cos. } x \dots (2);$$

hence this rule. To the logarithm of the arc reduced into seconds, with the decimal annexed, add the constant quantity 4.6855749, and from the sum subtract one third of the arithmetical complement of the log. cosine; the remainder will be the logarithmic sine of the given arc.

This rule will determine the log. sine of a very small arc with great accuracy; it was first given, without demonstration, by Dr. Maskelyne, in his Introduction to Taylor's Logarithms. The above proof is from Woodhouse's Trigonometry.

PROBLEM V.

To find the logarithmic tangent of a very small arc.

Let x be the arc; then, as we have found in last problem,

$$\sin. = x \cos. \frac{1}{3} x \therefore \frac{\sin. x}{\cos. x} = \tan. x = \frac{x}{\cos. \frac{2}{3} x}.$$

Hence, introducing the radius,

$$\log. \tan. x = \log. x + \frac{2}{3} (10 - \log. \cos. x).$$

The second member of this equation is equal to the second member of (1) in last problem, *plus* the arithmetical complement of $\log. \cos. x$; hence, since the second member of (2) is equivalent to the second member of (1), we have

$$\log. \tan. x = \log. n + 4.6855749 + \frac{2}{3} \text{ arith. comp. log. cos. } x \dots (3);$$

which furnishes this rule. To the logarithm of the arc reduced to seconds add the constant quantity 4.6855749, and two thirds of the arithmetical complement of the log. cosine, the sum is the log. tangent of the given arc.

PROBLEM V.

To find a small arc from its log. sine or its log. tangent.

1. Let the log. sine be given; then n being the number of seconds in the arc, the expression (2), in problem iv., gives

$$\begin{aligned}\log. n &= \log. \sin. x - 4.6855749 + \frac{1}{3} \text{arith. comp. log. cos. } x \\ &= \log. \sin. x + 5.3144251 - 10 + \frac{1}{3} \text{arith. comp. log. cos. } x;\end{aligned}$$

therefore, to find the arc from the log. sine the rule is this. To the log. sine of the small arc add 5.3144251, and $\frac{1}{3}$ of the arithmetical complement of the log. cosine; subtract 10 from the index of the sum, and the remainder will be the logarithm of the number of seconds in the arc.

2. Let the log. tangent be given; then from the expression (3), last problem, we have

$$\begin{aligned}\log. n &= \log. \tan. x - 4.6855749 - \frac{2}{3} \text{arith. comp. log. cos. } x \\ &= \log. \tan. x + 5.3144251 - 10 - \frac{2}{3} \text{arith. comp. log. cos. } x;\end{aligned}$$

that is, to the log. tangent of the small arc add 5.3144251, and from the sum subtract $\frac{2}{3}$ of the arithmetical complement of the log. cosine, take 10 from the index of the remainder, and we shall have the logarithm of the number of seconds in the arc.

Let us now apply each of the foregoing rules to an example.

1. Required the log. sine of $1^{\circ} 4' 48.754''$.

By the rule in problem iv. the process is as follows:

log. 64.8754	.	1.8120801
Constant No.	.	4.6855749
		6.4976550
$\frac{1}{3}$ arith. comp. log. cos.	.	0
log. sin. $1^{\circ} 4' 48.754''$	.	6.4976550.

By the tables the log. sine is found as follows:

log. sin. 1' 5"	.	6.4984882
log. sin. 1' 4"	.	6.4917548
Difference	.	.0067334

$$\therefore \log. \sin. 1' 4.8754'' = 6.4917548 + .8754 \times .0067334 \\ = 6.4976489.$$

2. Required the log. tangent of 7' 2.38".

By the Rule in Problem V.

log. 422.38	.	2.6257033
Constant No.	.	4.6855749
$\frac{2}{3}$ arith. comp. log. cos.		0
log. tan. 7' 2.38"	.	7.3112782.

By the Tables.

log. tan. 7' 3"	.	7.3119158
log. tan. 7' 2	.	7.3108879
		.0010279

$$\therefore \log. \tan. 7' 2.38'' = 7.3108879 + .38 \times .0010279 \\ = 7.3112785.$$

3. Required the arc whose log. sine is 6.4976550.

By the Rule, Problem V.

log. sine	.	6.4976550
Constant No.	.	5.3144251
$\frac{1}{3}$ arith. comp.	.	0
log. 64.8754	.	<u>1.8120801</u>

$\therefore$ the arc is $1' 4.8754''$.

By the Tables.

The proposed log. sine lies between log. sine $1' 4''$ and log. sine $1' 5''$, and the difference between these logs. is .0067334; also the difference between the proposed log. and log. sine $1' 4''$ is 59002; hence,

$$\text{required arc} = 1' 4'' + \frac{59002}{67334} = 1' 4.876''.$$

4. Required the arc whose log. tangent is 7.1644398.

By the Rule.

log. tan.	.	7.1644398
Constant No.	.	5.3144251
$\frac{2}{3}$ arith. comp.	.	<u>-3</u>
log. 301.2067	.	<u>2.4788646</u>

$\therefore$ the arc is $5' 1.2067''$.

By the Tables.

The proposed log. is between log. tan. $5' 1''$ and log. tan. $5' 2''$; the difference of these logs. is .0014404, and the difference of the proposed and log. tan. $5' 1''$ is .0002981.

$$\therefore \text{the arc is } 5' 1'' + \frac{2981}{14404} = 5' 1.2069''.$$

CHAPTER II.

INVESTIGATION OF EXPRESSIONS FOR THE SURFACE
OF A SPHERICAL TRIANGLE, AND FOR THE
SPHERICAL EXCESS.

(84.) It has been already shown (36) that two great circles always intersect in two points at the distance of a semicircle from each other. The space thus included by two great circles is called a *lune*, (see the fig. at page 66.)

The surface of a lune is to the surface of the whole sphere as the arc QQ' , or as the angle P of the lune, is to the whole circumference $IQHI$. This is pretty obvious, but it may be rigorously proved in the same way as it is proved in plane Geometry, that in the same circle any sector is to the whole circle as its arc is to the circumference, (Geom. prob. 23, Book 6). Hence, if we call the surface of the sphere S , and the angle of the lune ω degrees, the expression for its area will be

$S \frac{\omega}{360}$; or if, instead of degrees, ω represent the absolute length of

those degrees to radius 1, then the expression may be written $S \frac{\omega}{2\pi}$,

where π stands for the number 3.14159, &c.

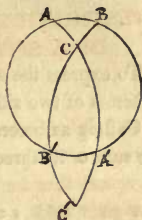
It can be proved, although not by the elementary principles of Trigonometry, that the surface of a sphere is equal to four times the area of one of its great circles;* that is, r being the radius of the sphere $S = 4\pi r^2$, so that the expression for the area of the lune is $2r^2\omega$. If we suppose r to be unity, the surface will be expressed by 2ω , that of the whole sphere being 4π .

* See "The Elements of the Integral Calculus," page 144.

PROBLEM I.

To express the area of a spherical triangle in terms of its three angles.

Let ABC be any spherical triangle, and produce the sides AC , BC , till they meet again in C' , forming the lune CC' . The triangle CAB will be a portion of an opposite lune equal to the lune CC' ; and this portion will obviously be equal to the portion $C'A'B'$, provided the arcs CA , CB , are equal to the arcs $C'A'$, $C'B'$. Now AA' is equal to CC' , each being a semicircle; hence, taking from each the common part CA' , we have $CA = C'A'$. In like manner $CB = C'B'$, and, therefore, the triangles ABC , $A'B'C'$, are equal. Hence the surface of the hemisphere, whose base is $ABA'B'$, is equal to the sum of the three lunes AA' , BB' , CC' , minus twice the triangle ABC ; that is, calling the surface of this triangle Σ .



$$\frac{1}{2} S = 2r^2(A + B + C) - 2\Sigma$$

$$\therefore \Sigma = r^2(A + B + C) - \frac{1}{4} S = r^2\{(A + B + C) - \pi\}.$$

where it must be observed that A , B , C , denote the lengths of the arcs which measure the angles of the proposed triangle to radius unity.

But, if we take A , B , C , and π in degrees, then since

$$180^\circ : \pi :: A + B + C - 180^\circ : \{A + B + C - 180^\circ\} \frac{\pi}{180^\circ}$$

the expression for Σ will be

$$\Sigma = r^2 \{A + B + C - 180^\circ\} \frac{\pi}{180^\circ} \dots (1).$$

If the radius of the sphere, on which the triangle is, be taken for unity, then calling the area in this case ϵ , we have

$$\epsilon = A + B + C - 180^\circ \dots (2);$$

which indicates that *the area of a triangle, on the surface of a sphere, whose radius is unity, is equal to the excess of its three angles above two right-angles*. This quantity is technically called the *spherical excess*, and the theorem (2) is known by the name of *Girard's theorem*.

It follows from this proposition that two spherical triangles are equal in surface, if the angles of the one are severally equal to those of the other, or, indeed, if the sum of the angles of the one triangle is equal to the sum of the angles of the other.

PROBLEM II.

To express the area of a spherical triangle, or the spherical excess in terms of two sides, and the included angle.

Calling as before the surface of the triangle to radius unity ϵ , and, the sum of its three sides s , we have, by last problem,

$$\epsilon = s - 180^\circ \therefore \cot. \frac{\epsilon}{2} = -\tan. \frac{1}{2} s.$$

But (27),

$$\tan. \frac{1}{2} s = \tan. \frac{1}{2} (A + B + C) = \frac{\tan. \frac{1}{2} (A + B) + \tan. \frac{1}{2} C}{1 - \tan. \frac{1}{2} (A + B) \tan. \frac{1}{2} C};$$

and, by Napier's analogy,

$$\tan. \frac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} (a + b)} \cot. \frac{1}{2} C;$$

hence, by substitution,

$$\cot. \frac{\epsilon}{2} = \frac{\cos. \frac{1}{2} (a - b) \cot. \frac{1}{2} C + \cos. \frac{1}{2} (a + b) \tan. \frac{1}{2} C}{\cos. \frac{1}{2} (a - b) - \cos. \frac{1}{2} (a + b)};$$

or multiplying the numerator by $2 \sin. \frac{1}{2} C \cos. \frac{1}{2} C$, and the denominator by its equal, $\sin. C$, (equa. 18, p. 55,)

$$\cot. \frac{\epsilon}{2} = \frac{2 \cos. \frac{1}{2} (a - b) \cos.^2 \frac{1}{2} C + 2 \cos. \frac{1}{2} (a + b) \sin.^2 \frac{1}{2} C}{\cos. \frac{1}{2} (a - b) \sin. C - \cos. \frac{1}{2} (a + b) \sin. C};$$

that is, by the formulas (1) and (2), page 46,

$$\cot. \frac{\epsilon}{2} = \frac{\cos. \frac{1}{2} a \cos. \frac{1}{2} b + \sin. \frac{1}{2} a \sin. \frac{1}{2} b (\cos.^2 \frac{1}{2} C - \sin.^2 \frac{1}{2} C)}{\sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. C};$$

but, (from 19, p. 55,)

$$\cos.^2 \frac{1}{2} C - \sin.^2 \frac{1}{2} C = \cos. C;$$

hence

$$\begin{aligned}\cot. \frac{\varepsilon}{2} &= \frac{\cot. \frac{1}{2} a \cot. \frac{1}{2} b + \cos. C}{\sin. C} \\ &= \left\{ \frac{\cot. \frac{1}{2} a \cot. \frac{1}{2} b}{\cos. C} + 1 \right\} \cot. C.\end{aligned}$$

To adapt this expression to logarithmic computation suppose first that $\cos. C$ is positive, and that we assume

$$\frac{\cot. \frac{1}{2} a \cot. \frac{1}{2} b}{\cos. C} = \tan.^2 \theta,$$

then

$$\cot. \frac{\varepsilon}{2} = \sec.^2 \theta \cot. C;$$

suppose, secondly, that $\cos. C$ is negative, then if $\frac{\cot. \frac{1}{2} a \cot. \frac{1}{2} b}{\cos. C}$ is numerically less than radius, assume it equal to $\sin.^2 \theta$, and we shall have

$$\cot. \frac{\varepsilon}{2} = \cos.^2 \theta \cot. C;$$

but if the same expression be numerically greater than radius, then assume it equal to $\sec.^2 \theta$, when we shall have

$$\cot. \frac{\varepsilon}{2} = \tan.^2 \theta \cot. C.$$

It may be remarked that, with the proposed data, the excess may be otherwise easily determined, by first finding, by the common formula, the third angle of the triangle, and then applying Girard's theorem.

PROBLEM III.

To determine the spherical excess when the three sides are given.

By formula 25, p. 57,

$$\cot. \frac{1}{2} a \cot. \frac{1}{2} b = \frac{1 + \cos. a}{\sin. a} \cdot \frac{1 + \cos. b}{\sin. b} = \frac{1 + \cos. a + \cos. b + \cos. a \cos. b}{\sin. a \sin. b}.$$

By formula (A), p. 74,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}.$$

By formulas (1), (2), p. 77,

$$2 \sin. \frac{1}{2} C \cos. \frac{1}{2} C = \sin. C = \frac{2}{\sin. a \sin. b} \sqrt{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a) \sin. \frac{1}{2} (S - b) \sin. (\frac{1}{2} S - c)}.$$

Substituting these values in the expression for $\cot. \frac{\epsilon}{2}$, last problem, we have

$$\cot. \frac{\epsilon}{2} = \frac{1 + \cos. a + \cos. b + \cos. c}{2 \sqrt{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a) \sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}}. \quad (2)$$

We may investigate another expression for the excess, as follows.

By the formulas (1), (2), page 77,

$$\sin. \frac{1}{2} A \cos. \frac{1}{2} B = \frac{\sin. (\frac{1}{2} S - b)}{\sin. c} \sqrt{\frac{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - c)}{\sin. b \sin. c}}$$

$$\sin. \frac{1}{2} B \cos. \frac{1}{2} A = \frac{\sin. (\frac{1}{2} S - a)}{\sin. c} \sqrt{\frac{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - c)}{\sin. b \sin. c}}.$$

By adding,

$$\sin. \frac{1}{2} (A + B) = \frac{\sin. (\frac{1}{2} S - b) + \sin. (\frac{1}{2} S - a)}{2 \sin. \frac{1}{2} c \cos. \frac{1}{2} c} \cos. \frac{1}{2} C.$$

By subtracting,

$$\sin. \frac{1}{2} (A - B) = \frac{\sin. (\frac{1}{2} S - b) - \sin. (\frac{1}{2} S - a)}{2 \sin. \frac{1}{2} c \cos. \frac{1}{2} c} \cos. \frac{1}{2} C.$$

But by formula (27), page 58,

$$\sin. (\frac{1}{2} S - b) + \sin. (\frac{1}{2} S - a) = 2 \sin. \frac{1}{2} c \cos. \frac{1}{2} (a - b).$$

$$\sin. (\frac{1}{2} S - b) - \sin. (\frac{1}{2} S - a) = 2 \cos. \frac{1}{2} c \sin. \frac{1}{2} (a - b).$$

Hence, by substitution,

$$\sin. \frac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} c} \cos. \frac{1}{2} C$$

$$\sin. \frac{1}{2} (A - B) = \frac{\sin. \frac{1}{2} (a - b)}{\sin. \frac{1}{2} c} \cos. \frac{1}{2} C.$$

Proceeding in the same way with the expressions for $\cos. \frac{1}{2} A \cos. \frac{1}{2} B$

and $\sin. \frac{1}{2}A \sin. \frac{1}{2}B$, there results

$$\cos. \frac{1}{2}(A - B) = \frac{\sin. \frac{1}{2}(a + b)}{\sin. \frac{1}{2}c} \sin. \frac{1}{2}C$$

$$\cos. \frac{1}{2}(A + B) = \frac{\cos. \frac{1}{2}(a + b)}{\cos. \frac{1}{2}c} \sin. \frac{1}{2}C.$$

Now, since

$$\sin. \frac{\epsilon}{2} = -\cos. \frac{1}{2}(A + B + C) = \sin. \frac{1}{2}(A + B) \sin. \frac{1}{2}C - \cos. \frac{1}{2}(A + B) \cos. \frac{1}{2}C,$$

we have, by substituting in the second member the first and last of the foregoing expressions,

$$\sin. \frac{\epsilon}{2} = \frac{\sin. \frac{1}{2}a \sin. \frac{1}{2}b}{\cos. \frac{1}{2}c} \sin. C;$$

or substituting for $\sin. C$ its value, as exhibited in last problem, and recollecting that (31)

$$\sin. a \sin. b = 4 \sin. \frac{1}{2}a \cos. \frac{1}{2}a \sin. \frac{1}{2}b \cos. \frac{1}{2}b,$$

we have

$$\sin. \frac{\epsilon}{2} = \sqrt{\frac{\sin. \frac{1}{2}S \sin. (\frac{1}{2}S - a) \sin. (\frac{1}{2}S - b) \sin. (\frac{1}{2}S - c)}{2 \cos. \frac{1}{2}a \cos. \frac{1}{2}b \cos. \frac{1}{2}c}} \dots (3);$$

an expression adapted to logarithms.

By combining the formulas (2) and (3) various others may be deduced. Thus, by multiplying them together, we have

$$\cos. \frac{\epsilon}{2} = \frac{1 + \cos. a + \cos. b + \cos. c}{4 \cos. \frac{1}{2}a \cos. \frac{1}{2}b \cos. \frac{1}{2}c} \dots (4).$$

But, formula (20), page 55,

$$\cos. \frac{1}{2}a = \sqrt{\frac{1}{2} + \frac{1}{2}\cos. a}, \cos. \frac{1}{2}b = \sqrt{\frac{1}{2} + \frac{1}{2}\cos. b}, \cos. \frac{1}{2}c = \sqrt{\frac{1}{2} + \frac{1}{2}\cos. c};$$

hence,

$$\cos. \frac{\epsilon}{2} = \frac{1 + \cos. a + \cos. b + \cos. c}{\sqrt{2(1 + \cos. a)(1 + \cos. b)(1 + \cos. c)}} \dots (5).$$

Squaring this,

$$2 \cos.^2 \frac{\epsilon}{2} = \frac{(1 + \cos. a + \cos. b + \cos. c)^2}{(1 + \cos. a)(1 + \cos. b)(1 + \cos. c)};$$

which, since $\cos. \epsilon = 2 \cos.^2 \frac{\epsilon}{2} - 1$, gives

$$\cos. \epsilon = \frac{(1 + \cos. a + \cos. b + \cos. c)^2 - (1 + \cos. a)(1 + \cos. b)(1 + \cos. c)}{(1 + \cos. a)(1 + \cos. b)(1 + \cos. c)} \dots (6);$$

also because $1 - \cos. \epsilon = \text{vers. } \epsilon$, we may change this into

$$\text{vers. } \epsilon = \frac{1 - \cos.^2 a - \cos.^2 b - \cos.^2 c + 2 \cos. a \cos. b \cos. c}{(1 + \cos. a)(1 + \cos. b)(1 + \cos. c)} \dots (7).$$

Lastly, by squaring the expression (3) and multiplying by 2, we have

$$2 \sin.^2 \frac{\epsilon}{2} = 1 - \cos. \epsilon = \text{vers. } \epsilon = \frac{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a) \sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}{2 \cos.^2 \frac{1}{2} a \cos.^2 \frac{1}{2} b \cos.^2 \frac{1}{2} c} \dots (8).$$

The expression, marked (2), is due to *De Gua*, as are those marked (4), (5), (6), and (7). The expression (3) is from *Cagnoli*, (*Trigon.* page 329.)

Since

$$\tan. \frac{\epsilon}{4} = \frac{1 - \cos. \frac{1}{2} \epsilon}{\sin. \frac{1}{2} \epsilon}.$$

we have, by combining the expressions (3) and (5),

$$\tan. \frac{\epsilon}{4} = \frac{1 - \cos.^2 \frac{1}{2} a - \cos.^2 \frac{1}{2} b - \cos.^2 \frac{1}{2} c + 2 \cos. \frac{1}{2} a \cos. \frac{1}{2} b \cos. \frac{1}{2} c}{\sqrt{\sin. \frac{1}{2} S \sin. (\frac{1}{2} S - a) \sin. (\frac{1}{2} S - b) \sin. (\frac{1}{2} S - c)}}$$

Now

$$1 - \cos.^2 \frac{1}{2} a - \cos.^2 \frac{1}{2} b = \sin.^2 \frac{1}{2} a - \cos.^2 \frac{1}{2} b,$$

$$\text{by equa. 5, p. 47,} \quad = \sin.^2 \frac{1}{2} a \sin.^2 \frac{1}{2} b - \cos.^2 \frac{1}{2} a \cos.^2 \frac{1}{2} b;$$

hence, the numerator of the above expression is equal to

$$\sin.^2 \frac{1}{2} a \sin.^2 \frac{1}{2} b - (\cos. \frac{1}{2} a \cos. \frac{1}{2} b - \cos. \frac{1}{2} c)^2,$$

which is the same as

$$\{\sin. \frac{1}{2} a \sin. \frac{1}{2} b + \cos. \frac{1}{2} a \cos. \frac{1}{2} b - \cos. \frac{1}{2} c\} \times$$

$$\{\sin. \frac{1}{2} a \sin. \frac{1}{2} b - \cos. \frac{1}{2} a \cos. \frac{1}{2} b + \cos. \frac{1}{2} c\};$$

or as

$$\{\cos. \frac{1}{2}(a-b) - \cos. \frac{1}{2}c\} \times \{\cos. \frac{1}{2}c - \cos. \frac{1}{2}(a+b)\} = (\text{by equ. 27, p. 58})$$

$$2 \sin. \frac{1}{2}(\frac{1}{2}S - b) \sin. \frac{1}{2}(\frac{1}{2}S - a) \times 2 \sin. \frac{1}{4}S \sin. \frac{1}{2}(\frac{1}{2}S - c).$$

Consequently, since (page 57)

$$\sqrt{\frac{1}{2} \tan. \frac{1}{2} A} = \frac{\sin. \frac{1}{2} A}{\sqrt{\sin. A}},$$

the foregoing expression for $\tan. \frac{\epsilon}{4}$ takes this very remarkable form, viz.

$$\tan. \frac{\epsilon}{4} = \sqrt{\tan. \frac{1}{4}S \tan. \frac{1}{2}(\frac{1}{2}S - a) \tan. \frac{1}{2}(\frac{1}{2}S - b) \tan. \frac{1}{2}(\frac{1}{2}S - c)};$$

which is *Lhuillier's* expression.

It follows from this problem that two spherical triangles are always equal in surface when the sides of the one are severally equal to those of the other, whether the triangles admit of coincidence or not.

PROBLEM IV.

Given the area of a spherical triangle on the surface of the earth in square feet, to determine the spherical excess.

Let the area of the triangle in feet be Σ , then, by problem I.,

$$\epsilon = \frac{\Sigma}{r^2} \cdot \frac{180^\circ}{3 \cdot 14159}.$$

Now the length of a degree, supposing the earth to be a perfect sphere, is 365154.6 feet; hence the earth's radius is $\frac{180}{3 \cdot 14159} \times 365154.6$ feet; consequently,

$$\epsilon = \frac{3 \cdot 14159 \Sigma}{180 \times (365154.6)^2} \text{ degrees;}$$

or if the excess be expressed in seconds, then

$$\epsilon = \frac{3 \cdot 14159 \Sigma \times 60^2}{180 (365154.6)^2} \text{ seconds.}$$

$$\therefore \log. \epsilon = \log. \Sigma + \log. 62.83185 \text{ \&c.} - 2 \log. 365154.6$$

$$= \log. \Sigma + 1.7981799 - 11.1249536$$

$$= \log. \Sigma - 9.3267737.$$

Hence, *from the logarithm of the area of the triangle in feet, subtract the constant logarithm 9.3267737, and the remainder will be the logarithm of the excess in seconds.*

This rule, which usually goes by the name of *General Roy's rule*, is in fact due to the late professor *Dalby*, by whom it was communicated to the General, when engaged with him in conducting the Trigonometrical Survey. (See the "Life of Mr. Dalby," in *Leybourn's Repository*, vol. v.)

By means of the rule just given we may very readily compute the spherical excess, provided that we previously know the area of the triangle in feet. In trigonometrical surveying, the triangle on the surface of the earth, comprised between any three stations, is necessarily so limited a portion of the whole sphere that its area, computed as a plane triangle from the measured data, cannot be affected with any error of consequence. On this hypothesis, therefore, the area of the triangle may be determined by one or other of the methods in prob. III., last chapter, and thence the excess ascertained by the above rule. Should the excess, thus deduced, exactly equal the excess of the three observed angles above two right-angles, we may be assured of the accuracy of the observations; but if they differ, the difference must be regarded as the amount of the errors with which the three observed angles are affected. If all of them were observed with equal care, so that there appear no reason why one should be more erroneous than another, the correction thus found must be distributed equally among them; but if it be suspected that one of the angles is less to be depended on than the others, then to this angle must be applied the greater part of the whole correction. The data being thus corrected, the required side or sides of the spherical triangle may be computed by the rules of spherical trigonometry; or the same object may be effected by plane trigonometry, with all requisite accuracy, provided we employ in the computation, not the corrected spherical angles, but these angles diminished each by one third of the spherical excess found as above, a truth which has been

established by Legendre, (See the Appendix to Brewster's translation of Legendre's Geometry.) Trigonometrical surveying is a very important application of the theory of trigonometry, but is too ample a subject to admit of being discussed in the present volume. The student will find a condensed account of these geodetical operations in the tenth section of Dr. Lardner's Trigonometry, and every requisite information in the *Géodésie* of M. Puissant and Colonel Mudge's account of the *Trigonometrical Survey of England and Wales*.*

Miscellaneous Expressions involving the Sides and Angles of a Spherical Triangle.

(85). We shall terminate the present chapter by the insertion of a few general expressions, involving the three sides and the three angles of a spherical triangle. Those formulas which have already been given in the second part of the work, are amply sufficient for the solution of every case in spherical trigonometry, but the sides and angles of a spherical triangle possess many other remarkable relations which are often called in aid in higher investigations concerning the sphere. A few of these, therefore, it may be proper to give. Let s represent half the sum of the sides a, b, c , and S , half the sum of the angles A, B, C , of a spherical triangle; then, by multiplying together the expressions for $\sin. \frac{1}{2}A$, $\cos. \frac{1}{2}A$, in art. (47), and those for $\sin. \frac{1}{2}a$, $\cos. \frac{1}{2}a$, in art. (49), and squaring the results, we have these equations;

$$\sin.^2 b \sin.^2 c \sin.^2 A = 4 \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c) = 4n^2. (1)$$

$$\sin.^2 B \sin.^2 C \sin.^2 a = 4 \cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C) = 4N^2. (2)$$

By multiplication,

$$\sin. a \sin. b \sin. c \sin. A \sin. B \sin. C = 4 N n \dots (3).$$

By division,

$$\frac{\sin. b}{\sin. B} \cdot \frac{\sin. c}{\sin. C} \cdot \frac{\sin. A}{\sin. a} = \frac{n}{N}.$$

* Some additional particulars respecting the spherical excess will be found in the supplement.

But the first two factors of this expression are each of them the reciprocal of the last.

$$\therefore \frac{\sin. b}{\sin. B} = \frac{\sin. c}{\sin. C} = \frac{\sin. a}{\sin. A} = \frac{n}{N} \dots \dots (4).$$

But from (1),

$$\frac{\sin. A}{\sin. a} = \frac{2n}{\sin. a \sin. b \sin. c} \therefore \frac{N}{n} = \frac{2n}{\sin. a \sin. b \sin. c} \dots \dots (5);$$

and from (2),

$$\frac{\sin. a}{\sin. A} = \frac{2N}{\sin. A \sin. B \sin. C} \therefore \frac{n}{N} = \frac{2N}{\sin. A \sin. B \sin. C} \dots \dots (6).$$

Substituting in (6) the value of N deduced from (5), and in (5) the value of n deduced from (6), we have, from the resulting equations, these expressions for n and N , viz.

$$n = \frac{1}{2} \{ \sin.^2 a \sin.^2 b \sin.^2 c \sin. A \sin. B \sin. C \}^{\frac{1}{3}} \dots \dots (7)$$

$$N = \frac{1}{2} \{ \sin.^2 A \sin.^2 B \sin.^2 C \sin. a \sin. b \sin. c \}^{\frac{1}{3}} \dots \dots (8);$$

and, for their ratio $\frac{n}{N}$, we have from these, as also from (4),

$$\frac{n}{N} = \left\{ \frac{\sin. a \sin. b \sin. c}{\sin. A \sin. B \sin. C} \right\}^{\frac{1}{3}} \dots \dots (9);$$

expressions which are remarkable for their symmetry.

Again, referring to the expressions for $\sin. \frac{1}{2}A$, and $\cos. \frac{1}{2}A$, at (47), we see that

$$\sin. \frac{1}{2} A \sin. \frac{1}{2} B \sin. \frac{1}{2} C = \frac{n^2}{\sin. s \sin. a \sin. b \sin. c} \dots \dots (10)$$

$$\cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C = \frac{n \sin. s}{\sin. a \sin. b \sin. c} \dots \dots (11)$$

$$\therefore \tan. \frac{1}{2} A \tan. \frac{1}{2} B \tan. C = \frac{n}{\sin.^2 s} \dots \dots (12).$$

And by referring to the expressions for $\sin. \frac{1}{2}a$, $\cos. \frac{1}{2}a$, at (49), we

see the truth of the following analogous equations; viz.

$$\sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c = \frac{-N \cos. S}{\sin. A \sin. B \sin. C} * \dots (13)$$

$$\cos. \frac{1}{2} a \cos. \frac{1}{2} b \cos. \frac{1}{2} c = \frac{N^2}{-\cos. S \sin. A \sin. B \sin. C} \dots (14)$$

$$\therefore \tan. \frac{1}{2} a \tan. \frac{1}{2} b \tan. \frac{1}{2} c = \frac{\cos.^2 S}{N} \dots (15).$$

From (10),

$$\begin{aligned} \sin. s &= \frac{n^2}{\sin. a \sin. b \sin. c \sin. \frac{1}{2} A \sin. \frac{1}{2} B \sin. \frac{1}{2} C} \dots \\ &=, \text{ by (5), } \frac{N}{2 \sin. \frac{1}{2} A \sin. \frac{1}{2} B \sin. \frac{1}{2} C} \dots (16). \end{aligned}$$

From (11),

$$\begin{aligned} \sin. s &= \frac{\sin. a \sin. b \sin. c \cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C}{n} \\ &=, \text{ by (5), } 2 \frac{n}{N} \cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C \dots (17); \end{aligned}$$

and, from (12),

$$\sin.^2 s = \frac{n}{\tan. \frac{1}{2} A \tan. \frac{1}{2} B \tan. \frac{1}{2} C} \dots (18).$$

In like manner, from (13), (14), (15) we have

$$\begin{aligned} \cos. S &= - \frac{\sin. A \sin. B \sin. C \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{N} \\ &=, \text{ by (6), } -2 \frac{N}{n} \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c \dots (19) \end{aligned}$$

$$\cos. S = - \frac{N^2}{\sin. A \sin. B \sin. C \cos. \frac{1}{2} a \cos. \frac{1}{2} b \cos. \frac{1}{2} c}$$

* This expression, as well as those marked 19, is usually given with an improper sign, viz. + instead of —, a mistake which seems to have arisen from confounding $\sqrt{\cos. S \cdot \cos. S}$ with $\sqrt{-\cos. S \times -\cos. S}$, which are, in fact, distinct expressions; the one being + cos. S, and the other — cos. S. See the chapter on *Imaginary Quantities*, in Young's Algebra.

$$=, \text{ by (6), } - \frac{n}{2 \cos. \frac{1}{2} a \cos. \frac{1}{2} b \cos. \frac{1}{2} c} \dots (20)$$

$$\cos.^2 S = N \tan. \frac{1}{2} a \tan. \frac{1}{2} b \tan. \frac{1}{2} c = \frac{N}{\cot. \frac{1}{2} a \cot. \frac{1}{2} b \cot. \frac{1}{2} c} \dots (21).$$

(86). In addition to these we shall here put down a few other useful expressions immediately deducible from the four equations which we had occasion to investigate at p. 194; and which are as follows:

$$\sin. \frac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} C}{\cos. \frac{1}{2} c} \cos. \frac{1}{2} (a - b) \dots (22)$$

$$\sin. \frac{1}{2} (A - B) = \frac{\cos. \frac{1}{2} C}{\sin. \frac{1}{2} c} \sin. \frac{1}{2} (a - b) \dots (23)$$

$$\cos. \frac{1}{2} (A + B) = \frac{\sin. \frac{1}{2} C}{\cos. \frac{1}{2} c} \cos. \frac{1}{2} (a + b) \dots (24)$$

$$\cos. \frac{1}{2} (A - B) = \frac{\sin. \frac{1}{2} C}{\sin. \frac{1}{2} c} \sin. \frac{1}{2} (a + b) \dots (25).$$

From these equations we immediately deduce the following analogous ones, viz.

$$\sin. \frac{1}{2} (a + b) = \frac{\sin. \frac{1}{2} c}{\sin. \frac{1}{2} C} \cos. \frac{1}{2} (A - B) \dots (26)$$

$$\sin. \frac{1}{2} (a - b) = \frac{\sin. \frac{1}{2} c}{\cos. \frac{1}{2} C} \sin. \frac{1}{2} (A - B) \dots (27)$$

$$\cos. \frac{1}{2} (a + b) = \frac{\cos. \frac{1}{2} c}{\sin. \frac{1}{2} C} \cos. \frac{1}{2} (A + B) \dots (28)$$

$$\cos. \frac{1}{2} (a - b) = \frac{\cos. \frac{1}{2} c}{\cos. \frac{1}{2} C} \sin. \frac{1}{2} (A + B) \dots (29).$$

From (22) and (23) we have,

$$\sin.^2 \frac{1}{2} (A + B) \cos.^2 \frac{1}{2} c = \cos.^2 \frac{1}{2} C \cos.^2 \frac{1}{2} (a - b)$$

$$\sin.^2 \frac{1}{2} (A - B) \sin.^2 \frac{1}{2} c = \cos.^2 \frac{1}{2} C \sin.^2 \frac{1}{2} (a - b).$$

Hence, by addition,

$$\sin.^2 \frac{1}{2} (A - B) \sin.^2 \frac{1}{2} c + \sin.^2 \frac{1}{2} (A + B) \cos.^2 \frac{1}{2} c = \cos.^2 \frac{1}{2} C \dots (30).$$

In like manner, from (24) and (25),

$$\cos.^2 \frac{1}{2} (A - B) \sin.^2 \frac{1}{2} c + \cos.^2 \frac{1}{2} (A + B) \cos.^2 \frac{1}{2} c = \sin.^2 \frac{1}{2} C . . (31).$$

Again, from (26) and (27), we have

$$\sin.^2 \frac{1}{2} (a + b) \sin.^2 \frac{1}{2} C = \sin.^2 \frac{1}{2} c \cos.^2 \frac{1}{2} (A - B)$$

$$\sin.^2 \frac{1}{2} (a - b) \cos.^2 \frac{1}{2} C = \sin.^2 \frac{1}{2} c \sin.^2 \frac{1}{2} (A - B);$$

and, by addition,

$$\sin.^2 \frac{1}{2} (a - b) \cos.^2 \frac{1}{2} C + \sin.^2 \frac{1}{2} (a + b) \sin.^2 \frac{1}{2} C = \sin.^2 \frac{1}{2} c . . . (32);$$

and, in like manner, from (28) and (29), we get

$$\cos.^2 \frac{1}{2} (a - b) \cos.^2 \frac{1}{2} C + \cos.^2 \frac{1}{2} (a + b) \sin.^2 \frac{1}{2} C = \cos.^2 \frac{1}{2} c . . . (33).$$

CHAPTER III.

ON THE RELATIONS BETWEEN THE CORRESPONDING VARIATIONS OF THE PARTS OF A TRIANGLE.

In the present chapter we propose briefly to examine into the effect produced on the sides and angles of a triangle, by a small change taking place in the magnitude of one of them; that is, to estimate the amount of error affecting any part which may have been determined from data, not strictly accurate, and thence to ascertain under what circumstances a small inaccuracy in a proposed datum will least affect the accuracy of the result. This becomes a very essential matter of enquiry in all the more delicate practical operations of trigonometry, because, since the data furnished by observation necessarily fall short of strict accuracy, on account of the imperfection of instruments, and other unavoidable defects, we ought to know under what circumstances our observation should be made, so that the small error with which it

is affected may have the least possible influence on the quantity to be determined from it. The following problems will sufficiently show the method of arriving at this knowledge.

PROBLEM I.

In a right-angled plane triangle, whose base is b and altitude a , it is required to determine the error committed in calculating a , by means of the given base b , and the observed angle opposite to a .

Let us consider a to represent the true angle opposite to a , from which that given by observation varies by a small quantity, which we shall represent by δa , and call *the variation of a* , then the sought side, which would be given by the equation $a = b \tan. a$, is affected by an error δa , so that instead of a it is $a + \delta a$, and this we determine from the equation

$$a + \delta a = b \tan. (a + \delta a);$$

in which, by subtracting the preceding equation, we find the value of δa to be

$$\begin{aligned} \delta a &= b \{ \tan. (a + \delta a) - \tan. a \} \\ &= \frac{b \sin. \delta a}{\cos. a \cos. (a + \delta a)} \text{ (art. 27).} \end{aligned}$$

Now, by hypothesis, δa is very small, so that we may substitute it for its sine, and $\cos. a$ instead of $\cos. (a + \delta a)$,

$$\therefore \delta a = \frac{b \delta a}{\cos.^2 a};$$

in which expression δa is the length of the arc to radius 1, which measures the angular error.

To determine what length b must be to render the variation δa , the least possible under the same amount of error δa in a , we have

$$\begin{aligned} b &= a \cot. a \therefore \delta a = \frac{a \cot. a \delta a}{\cos.^2 a} \\ &= \frac{a \delta a}{\sin. a \cos. a} = 2 \frac{a \delta a}{\sin. 2a}; \end{aligned}$$

hence δa will be the least possible when $\sin. 2a$ is the greatest possible, that is, when $a = 45^\circ$: so that in order to determine the height of a tower or steeple, &c. with the utmost accuracy, by means of an observation of its angular altitude, we should make the observation at a distance from the object as nearly as possible equal to its height.

PROBLEM II.

In a right-angled spherical triangle is given one of the oblique angles to determine the variation of the opposite side, arising from a small variation of the hypotenuse.

Let A be the constant angle, a its opposite side, and c the hypotenuse; then

$$\sin. a = \sin. A \sin. c$$

$$\sin. (a + \delta a) = \sin. A \sin. (c + \delta c)$$

$\therefore$ by subtraction,

$$\sin. (a + \delta a) - \sin. a = \sin. A \{ \sin. (c + \delta c) - \sin. c \};$$

that is, (page 58, equa. 27,)

$$2 \cos. (a + \frac{1}{2} \delta a) \sin. \frac{1}{2} \delta a = 2 \sin. A \cos. (c + \frac{1}{2} \delta c) \sin. \frac{1}{2} \delta c$$

$$\therefore \sin. \frac{1}{2} \delta a = \frac{\sin. A \cos. (c + \frac{1}{2} \delta c) \sin. \frac{1}{2} \delta c}{\cos. (a + \frac{1}{2} \delta a)};$$

and if $\delta a, \delta c$, be very small,

$$\delta a = \frac{\sin. A \cos. c}{\cos. a} \delta c;$$

or, substituting for $\sin. A$ its value from the first equation,

$$\delta a = \frac{\sin. a}{\cos. a} \cdot \frac{\cos. c}{\sin. c} \delta c = \tan. a \cot. c \delta c;$$

which variation will be the least possible when $\cot. c$ is least, or when $c = 90^\circ$. It would seem from the expression for δa , that in this case δa is absolutely 0, which we know cannot be. Indeed, no result deduced like that above, from a process in which certain small quantities

are rejected, can be considered as perfectly accurate, although they may approximate so nearly to the truth as to be practically admissible as such. If we restore the $\frac{1}{2}\delta c$ which has been neglected, and write the above result thus,

$$\delta a = \tan. a \cot. (c + \frac{1}{2}\delta c) \delta c;$$

then, in the case of $c = 90^\circ$, the expression becomes

$$\delta a = -\tan. a \tan. \frac{1}{2}\delta c \cdot \delta c;$$

or, considering the very small arc $\frac{1}{2}\delta c$ to be equal to its tangent, we have in the case supposed

$$\delta a = -\frac{1}{2}\tan. a (\delta c)^2,$$

the same expression as other wise determined by *Professor Airy* in his *Treatise on Trigonometry*, in the *Encyclopædia Metropolitana*.

PROBLEM III.

In an oblique-angled spherical triangle are given two sides to determine the variation produced in the third side by a small variation of the opposite angle.

Let a, b , be the two given sides, C the included angle, and c the side opposite to it. Then

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C$$

$$\cos. (c + \delta c) = \cos. a \cos. b + \sin. a \sin. b \cos. (C + \delta C);$$

$\therefore$ by subtraction,

$$\cos. (c + \delta c) - \cos. c = \sin. a \sin. b \{ (C + \delta C) - \cos. C \};$$

that is,

$$2 \sin. (c + \frac{1}{2}\delta c) \sin. \frac{1}{2}\delta c = 2 \sin. a \sin. b \sin. (C + \frac{1}{2}\delta C) \sin. \frac{1}{2}\delta C.$$

Hence, if δC be very small,

$$\sin. c \delta c = \sin. a \sin. b \sin. C \delta C$$

$$\therefore \delta c = \frac{\sin. a \sin. b \sin. C}{\sin. c} \delta C$$

$$= \sin. a \sin. b \delta C;$$

and δc is therefore the least possible when $\sin. C$ is the least possible, that is, when $C = 0$. To find the expression for δc , in this case, restore what has been rejected, and we shall have

$$\delta c = \frac{\sin. a \sin. b \sin. (C + \frac{1}{2} \delta C)}{\sin. c} \delta C;$$

which, when $C = 0$, and $\frac{1}{2} \delta C$ very small, becomes

$$\delta c = \frac{\sin. a \sin. b}{2 \sin. c} (\delta C)^2.$$

PROBLEM IV.

In an oblique-angled spherical triangle are given, as before, two sides and the included angle, to find the variation produced in one of the opposite angles by a small variation in the included angle.

Let a, b , be the given sides, C the included angle, then we have to find what influence a small variation in the value of the angle C will have on the angle A opposite a . For this purpose we shall deduce a suitable formula, as follows: substitute the expression for $\cos. c$, on the opposite page, in the corresponding expression for $\cos. a$, and we shall have the equation

$$\cos. A \sin. c = \cos. a \sin. b - \sin. a \cos. b \cos. C;$$

$$\therefore \cos. A \frac{\sin. c}{\sin. a} = \cot. a \sin. b - \cos. b \cos. C.$$

But

$$\frac{\sin. c}{\sin. a} = \frac{\sin. C}{\sin. A};$$

hence, by substitution,

$$\cot. A \sin. C = \cot. a \sin. b - \cos. b \cos. C$$

$$\cot. (A + \delta A) \sin. (C + \delta C) = \cot. a \sin. b - \cos. b \cos. (C + \delta C);$$

and, by subtraction,

$$\cot. (A + \delta A) \sin. (C + \delta C) - \cot. A \sin. C = \cos. b \{ \cos. C - \cos. (C + \delta C) \}.$$

The first side of this equation is the same as
 $\cot. (A + \delta A) \{ \sin. (C + \delta C) - \sin. C \} + \sin. C \{ \cot. (A + \delta A) - \cot. A \};$
 and the quantities within the brackets are respectively the same as

$$2 \cos. (C + \frac{1}{2} \delta C) \sin. \frac{1}{2} \delta C \text{ and } \frac{-\sin. \delta A}{\sin. A \sin. (A + \delta A)}.$$

Also, the second side of the same equation is the same as

$$\cos. b \cdot 2 \sin. (C + \frac{1}{2} \delta C) \sin. \frac{1}{2} \delta C;$$

consequently,

$$2 \cot. (A + \delta A) \cos. (C + \frac{1}{2} \delta C) \sin. \frac{1}{2} \delta C - \frac{\sin. C \sin. \delta A}{\sin. A \sin. (A + \delta A)} =$$

$$2 \cos. b \sin. (C + \frac{1}{2} \delta C) \sin. \frac{1}{2} \delta C;$$

and, therefore, when δC and δA are very small,

$$\cot. A \cos. C \delta C - \frac{\sin. C}{\sin.^2 A} \delta A = \cos. b \sin. C \delta C$$

$$\therefore \delta A = \frac{\sin.^2 A}{\sin. C} (\cot. A \cos. C - \cos. b \sin. C) \delta C.$$

The foregoing examples are those selected by *Professor Airy* in his *Treatise on Trigonometry*, before referred to, and we have here adopted his processes. But the instrument of investigation generally the best adapted to inquiries of this kind is the *Differential Calculus*.

END OF THE TRIGONOMETRY.

CHAPTER I.
ON SPHERICAL GEOMETRY.
SUPPLEMENT.

ON SPHERICAL GEOMETRY, POLAR TRIANGLES,
&c.

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ON SPHERICAL GEOMETRY, POLAR TRIANGLES, &c.

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CHAPTER I.

ON SPHERICAL GEOMETRY.

IN the commencement of the Spherical Trigonometry, a small collection of propositions, such as were necessary in the character of fundamental principles upon which to build the subsequent analytical investigations, was given. At the request of the Author, we here propose to add a few others, and shall endeavour to select such as may serve the double purpose of facilitating our future inquiries, and of interesting the mind of the student in some of the most beautiful classes of Geometrical research that are yet known to exist; we shall commence with a few properties analogous to the more elementary propositions in Euclid, and which are very often *assumed* by writers in spherical trigonometry, both unnecessarily and improperly.

1. Let O be the spherical centre of a circle, and AB any great circle chord: the perpendicular* OK demitted from the centre upon AB will bisect it.

Draw AO, BO. Then from the right-angled triangles AKO, BKO, we have

$$\frac{\cos. OA}{\cos. OB} = \frac{\cos. AK \cos. OK}{\cos. BK \cos. OK} = \frac{\cos. AK}{\cos. BK}.$$

But,

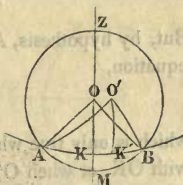
$$OA = OB, \text{ and } \therefore AK = BK.$$

2. Conversely, if OK bisect AB, it will cut it at right angles.

For

$$\cos. AKO = \frac{\cos. AO - \cos. AK \cos. KO}{\sin. AK \sin. KO}$$

* Always meaning a great circle perpendicular, except expressly stated otherwise.



$$\cos. BKO = \frac{\cos. BO - \cos. BK \cos. KO}{\sin. BK \sin. KO}.$$

But the right-hand sides of these questions are equal term for term, and therefore

$$\cos. AKO = \cos. BKO, \text{ or}$$

$$AKO = BKO;$$

and as AK, KB are one great circle, the angles at K are right angles: the tenth definition of the first book of Euclid applying to spherical as well as to plane angles.

3. If the great circle chord AB, be bisected at right angles at K, by the great circle ZM, this perpendicular shall pass through the centre of the circle.

For, assume for a moment that the centre is at O', not in the circle ZM; and draw the perpendicular O'K'. Now, we have seen that O'K' bisects AB in K' when O' is the centre, or that

$$AK + KK' = BK - KK';$$

But, by hypothesis, $AK = BK$, and therefore, subtracting the latter equation,

$$KK' = -KK',$$

which is only true when K, K' coincide, that is, when O'K' coincides with OK, or when O' is in ZM. The centre is therefore in ZM.

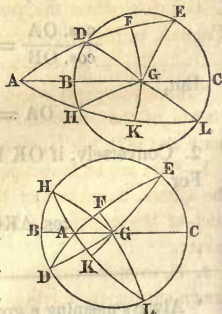
4. If two great circles which cut one another at A, be intersected by a circle of the sphere in D, E, and H, L respectively, the rectangles of the tangents of the semi-segments into which they are divided shall be equal.

That is,

$$\tan. \frac{1}{2} AE \tan. \frac{1}{2} AD = \tan. \frac{1}{2} AH \tan. \frac{1}{2} AL.$$

For find the centre G, draw AG meeting the circle in B and C, draw the perpendiculars GF, GK, and join GD, GE, GH, GL.

Then,



$$\cos. AK \cos. KG = \cos. AG = \cos. AF \cos. FG$$

$$\cos. LK \cos. KG = \cos. GL = \cos. FE \cos. FG.$$

From these we have, by subtraction, addition, and subsequent division.

$$\frac{(\cos. AK - \cos. KL)}{(\cos. AK + \cos. KL)} = \frac{(\cos. AF - \cos. FE)}{(\cos. AF + \cos. FE)};$$

and hence, by dividing (28) by (29), page 58,

$$\tan. \frac{1}{2}(AK - KL) \tan. \frac{1}{2}(AK + KL) = \tan. \frac{1}{2}(AF - FE) \tan. \frac{1}{2}(AF + FE),$$

that is,

$$\tan. \frac{1}{2} AH \tan. \frac{1}{2} AL = \tan. \frac{1}{2} AD \tan. \frac{1}{2} AE.$$

The analogue to Euclid III. 35, may be seen in another form in the Math. Repository, No. 23; part II., p. 131, 2.

5. Let the secants in the case where A is without the circle take the position of tangents. Then D, E, F, coalesce, and so do H, K, L. Then the equation just obtained becomes

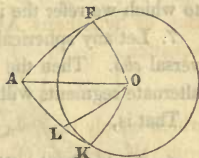
$$\tan.^2 \frac{1}{2} AF = \tan.^2 \frac{1}{2} AK,$$

or

$$AF = AK.$$

The tangents from any points to the circle are therefore equal.

The case when the point is within the circle is demonstrated by *Cagnoli*, in his Trigonometry, but the other case he has not noticed.*



* Since this paper was written, *Professor Lowry* has sent me the enunciation of the proposition in the text, accompanied by the foregoing remark and with the following corollaries subjoined, viz.

1. If an arc be drawn perpendicular to the diameter of a small circle of the sphere, the square of the tangent of half this arc will equal the product of the tangents of half the segments into which it divides the diameter.

2. If, from the extremities of the diameter of the small circle, arcs be drawn cutting the circle in the same point as the perpendicular, then the sum of the squares of the sines of half these arcs will equal the square of the sine of half the diameter of the small circle.

6. We may easily prove, also, that the great circle drawn through K at right angles to the radius, OK, touches the circle.

For draw any other arc from O, as OL.

Then, because K is a right-angle, we have

$$\cos. LK \cos. KO = \cos. LO.$$

But $\cos. LK < 1$, and therefore $\cos. LO < \cos. KO$, or

$$LO > KO;$$

and L will therefore fall *without* the circle; or, no part of KA falls within the circle whence KA is a tangent.

It is unnecessary to dwell at greater length upon these simple subjects; the nature of the inquiry and the method of pursuing it, as well as its close analogy to the corresponding properties in the Elements of Euclid, must be at once apparent.

We may add that in the *Repository*, as above referred to, some other remarkable analogies to plain properties are derived by similar methods, to which we refer the inquiring reader.

7. Let any spherical triangle be cut by a transversal *cba*. Then the products of the sines of the alternate segments will be equal.

That is,

$$\sin. Ac \sin. Ba \sin. Cb = \sin. cB \sin. aC \sin. bA.$$

For

$$\sin. Ac : \sin. Ab :: \sin. \angle b : \sin. \angle c$$

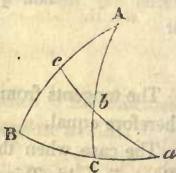
$$\sin. aB : \sin. Bc :: \sin. \angle c : \sin. \angle a$$

$$\sin. Cb : \sin. Ca :: \sin. \angle a : \sin. \angle b.$$

Hence, multiplying, and effacing the common terms in the antecedent and consequent of the result, we have the properties stated above.

8. The student can show, *ex absurdo*, that if the equality above stated takes place, the three points *a, b, c*, are in the same great circle.

9. If through any point P on the surface of the sphere three great circles be described, which also pass through the angles of the triangle ABC, and cut the opposite sides in *a, b, c*, respectively, then

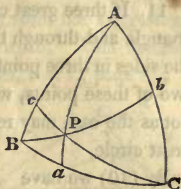


$$\sin. Aa \sin. Ba \sin. bC = \sin. aC \sin. Bc \sin. bA.$$

For the two spherical triangles BaA and CaA cut by the two transversals Cc , Bb , give respectively

$$\sin. AP \sin. aB \sin. bC = \sin. aP \sin. Bc \sin. bA$$

$$\sin. aP \sin. cB \sin. cA = \sin. AP \sin. aC \sin. cB;$$



which multiplied, and the common terms effaced, give the enunciated property.

10. If through any point P in a given great circle Aa , which passes through an angle, of a spherical triangle, great circles be drawn to the remaining angles cutting the opposite sides in b , c , respectively, then the great circle bc will always pass through the same point a' in the great circle BC , and so divide it that

$$\sin. Ba : \sin. ac :: \sin. Ba' : \sin. a' C.$$

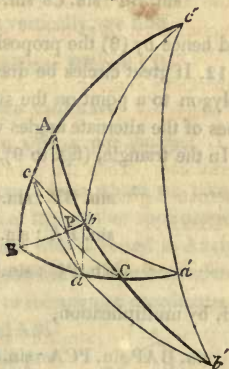
For by (9 and 7) we have, respectively,

$$\sin. Ba : \sin. aC :: \sin. Bc \sin. Ab :$$

$$\sin. cA \sin. bC$$

$$\sin. Ba' : \sin. a' C : \sin. Bc \sin. Ab :$$

$$\sin. cA \sin. bC ;$$



when, by equality of ratios, we have

$$\sin. Ba : \sin. aC :: \sin. Ba' : \sin. a' C.*$$

* This division of an arc is analogous to that which in *plano* is called the *harmonical* division of a line. Some of the most interesting properties of elementary geometry flow from considerations respecting the mode of division; and the spherical properties have perfect analogies to those. A few of these may be seen in the paper above mentioned in the Repository; and others will appear in a future number. Some curious investigations on this subject, by Professor Lowry, may also be seen in vol. II., new series of the same work, quest. 223. His processes, however, are totally different from those just adverted to.

11. If three great circles be drawn through the angles of a spherical triangle and through the same point on the surface of the sphere, cutting the sides in three points; three other great circles, each passing through two of these points, will intersect the sides of the triangle (produced or not as the case may require,) in the circumference of one and the same great circle,

By (10) we have

$$\sin. Ba : \sin. aC :: \sin. Ba' : \sin. a'C$$

$$\sin. Cb : \sin. bA :: \sin. Cb' : \sin. b'A$$

$$\sin. cA : \sin. cB :: \sin. c'A : \sin. c'B;$$

or, by multiplying vertically, and bearing (9) in mind,

$$\sin. Ba' \cdot \sin. Cb' \sin. c'A = \sin. a'C \cdot \sin. b'A \cdot \sin. c'B;$$

and hence by (8) the proposition is established.*

12. If great circles be drawn from the angular points of any spherical polygon to a point on the surface of the sphere, the product of the sines of the alternate angles will be equal.

In the triangle, (fig. to 9),

$$\sin. BP : \sin. PA :: \sin. BAP : \sin. ABP$$

$$\sin. PA : \sin. PC :: \sin. PCA : \sin. PAC$$

$$\sin. PC : \sin. PB :: \sin. PBC : \sin. PCB;$$

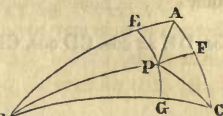
and, by multiplication,

$$\sin. BAP \sin. PCA \cdot \sin. CBP = \sin. ABP \sin. PAC \sin. PCB.$$

* This remarkable proposition appears to have been discovered by Carnot, and was first published by him in 1803, and afterwards in 1806, in the *Geometry of Position*, and the *Essay upon Transversals*. It was subsequently and independently discovered by an eminent mathematician in this country, Mr. Whitley, who inferred it from the corresponding plane one, in the *Ladies' Diary*, 1817. The demonstration above given is taken entirely from Carnot, and it is a beautiful model for the method of conducting such inquiries. More ample information on these subjects may be had in the Repository, *ub supra*.

The student is required to prove it for four, five, &c. sided figures, and is recommended to complete the argument from the suggestions furnished by the particular cases of the general truth. This theorem is due to *Professor Lowry*, Math. Rep. old series, vol. i., page 90.

Let ABC be a spherical triangle, and P a point on the surface of the sphere, from which perpendiculars PE , PF , PG , are drawn to the sides of the triangle: then the product of the cosines of the alternate segments will be equal to one another. For



$$\cos. AE \cos. EP = \cos. AP = \cos. AF \cos. FP$$

$$\cos. BG \cos. GP = \cos. BP = \cos. BE \cos. EP$$

$$\cos. CF \cos. FP = \cos. PC = \cos. CG \cos. GP.$$

and multiplying the first and last columns vertically, we find

$$\cos. AE \cos. BG \cos. CF = \cos. AF \cos. BE \cos. CG.$$

Cor. If the triangle in *triquadrantal*, we shall have

$$\tan. AE \tan. BG \tan. CF = 1 = \cot. AF \cot. BE \cot. CG.$$

14. We shall here give (although we forget from whom we take it, and what kind of demonstration was given of it,) another such property of the triquadrantal triangle; and the student who is versed in Analytical Geometry, will recognise in it the trigonometrical demonstration of a remarkable property of a point referred to rectangular coordinates.

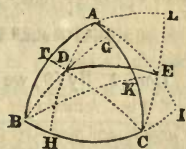
Let D , E be two points on the sphere, and ABC a triquadrantal triangle. Then we have this property, viz.

$$\cos. DE = \cos. DA \cos. AE + \cos. DB \cos. BE + \cos. DC \cos. CE.$$

For,

$$\cos. DE = \cos. CD \cos. CE + \sin. CD \sin. CE \cos. DCE \dots (a);$$

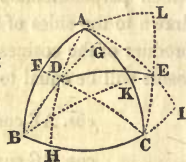
and, by right-angled triangles,



$$\sin. CD = \frac{\cos. AD}{\cos. AF} \text{ and } \sin. CE = \frac{\cos. AE}{\cos. AL} \dots (b).$$

the angle DCE is measured by $FL = AF \pm AL$; and hence (a) becomes

$$\begin{aligned} \cos. DE &= \cos. CD \cos. CE + \frac{\cos. AD}{\cos. AF} \cdot \frac{\cos. AE}{\cos. AL} \{ \cos. AF \cos. AL + \\ &\quad \sin. AF \sin. AL \} \\ &= \cos. CD \cos. CE + \cos. AD \cos. AE \mp \\ &\quad \frac{\cos. AD \cos. AE \sin. AF \sin. AL}{\cos. AF \cos. AL} \dots (c). \end{aligned}$$



But, by right-angled triangles,

$$\cos. AD = \cos. AF \cos. FD, \text{ and } \cos. AE = \cos. AL \cos. LE;$$

also,

$$\sin. AF = \cos. BF, \text{ and } \mp \sin. AL = \cos. BL.$$

hence the last term of (c) reduces to

$$\cos. FD \cos. BF \cos. LE \cos. BL,$$

and by right-angled triangles, the first pair of these factors is equal to $\cos. BD$, and the second pair to $\cos. BE$, and thus is the proposed theorem established.*

Cor. 1. When D and E coincide, $\cos. DE = 1$, and we have

$$\cos.^2 AD + \cos.^2 BD + \cos.^2 CD = 1$$

$$\sin.^2 DH + \sin.^2 DG + \sin.^2 DF = 1.$$

Cor. 2. By (9) we have

* It may be proper to mention here, that since the above demonstration was written I have remarked the same property in *Dr. Luby's Trigonometry*, p. 61-2; but I must have first met with it elsewhere, as I well recollect that it was unaccompanied with any proof. *Dr. Luby's* demonstration is a good deal similar to mine.

$$\begin{aligned}\sin. AF \sin. BH \sin. CG &= \sin. BF \sin. AG \sin. CH \\ &= \cos. AF \cos. BH \cos. CG\end{aligned}$$

or, by division,

$$\tan. AF \tan. BH \tan. CG = 1 = \tan. AL \tan. BL \tan. CK.$$

Cor. 3. When $DE = \frac{\pi}{2}$ we have

$$\cos. DA \cos. AE + \cos. DB \cos. BE + \cos. DC \cos. CE = 0;$$

(vide Young's Anal. Geom., p. 227.)

15. The following propositions, dependent upon what has been done here, or else upon similar methods, are left as exercises for the student.

(a.) Let a transversal great circle cut any spherical polygon; dividing each side into two segments; the product of the sines of the one set of alternate segments will be equal to the product of those of the other set.

✓ (b.) If a great circle bisect the angle of a triangle, (either interior or exterior,) the sines of the segments of the base have the same ratio as the sines of the sides.

(c.) The three bisectors pass through the same point on the sphere.

(d.) Perpendiculars to the middles of the sides pass through the same point, (centre of circumscribing circle).

(e.) Perpendiculars from the angles of the triangle to the opposite sides pass through the same point.

(f.) Great circles joining the middles of the sides to the opposite angles intersect in the same point.

(g.) Great circles joining the points of contact of the inscribed circles with the sides, and the opposite angles pass through the same point.

(h.) Great circles passing through the points of contact of the circle which touches a triangle exteriorly, and the opposite angles, pass through the same point.

(i.) Great circles bisecting the interior angles of a spherical triangle meet the opposite sides in three points, which are situated in one great circle of the sphere.

(k.) Show under what conditions the propositions (12) and (13) admit of conversion.

(l.) Perpendicular from the angles of a triangle upon the opposite

sides intersect in three points, and the triangle formed by joining these points has its angles bisected by the said perpendiculars.

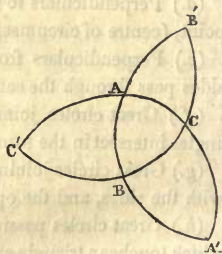
It would have been easy to extend and to vary these subjects almost without limit. As the method of Transversals is the most powerful one yet known for the investigation of spherical determinate theorems, (seeming to make up for the deficiency of parallels and similar triangles, the great organon in plane researches,) we thought it better to dwell upon this sufficiently to give the student a real insight into the character of its processes and to furnish him with a few suitable exercises for his own improvement in such investigations.*

CHAPTER II.

ASSOCIATED TRIANGLES.

1. Let ABC be a spherical triangle, having its sides produced to meet again in $A'B'C'$, respectively opposite to the angles A, B, C .

Four triangles are thus formed which have a necessary relation to one another. These we propose to call *the associated system of triangles*, or simply *the associated triangles*.



* A number of important properties of spherical triangles, demonstrated geometrically, by Professr Lowry, may be seen in the first vol. of the old series of the Mathematical Repository, and some others in Howard's Spherical Geometry, 1798. The subject, however, is still open to indefinite research, and offers ample reward to those whose taste may lead them to cultivate it. See also note D.

That which was first drawn (ABC), and which serves as a basis of the rest, we call the *fundamental triangle of the associated system*, or simply *the fundamental triangle*.

The others, two sides of each being supplements of two sides of the fundamental, and two angles of two angles, we call the *supplemental triangles of the associated system*, or simply the *supplemental triangles*.*

Moreover, when we wish to specify any one of the supplemental triangles, we shall do it by reference to the side in it which is common to a side, or an angle which is equal to an angle of the fundamental triangle. Thus, to designate the triangles $BA'C$, $CB'A$, $AC'B$, we say the supplemental triangle taken with respect to A' , (or a , as the case may be,) with respect to B' , or with respect to C' .

As a uniformity of notation is essential in inquiries like these, related to classes of similar objects, we shall attempt to conform to the established notation as a basis. Thus, abc are the sides of ABC ,

$$a \quad b, \quad c, \quad . \quad . \quad BA'C$$

$$a, \quad b \quad c, \quad . \quad . \quad CB'A$$

$$a, \quad b, \quad c \quad . \quad . \quad AC'B.$$

In which the number of subscribed accents points out the particular triangle designated, considering them to be ranged round the fundamental one in the order of the letters A' , B' , C' .

* The term "*supplemental*" has been used by English Mathematicians to designate that triangle which is now universally denominated the "*polar triangle*." The word has ceased to be used in that sense for some years, and as it is so peculiarly adapted to express the triangles which are formed by producing the sides of the fundamental, we have not hesitated to adopt it. We give this notice, however, of the change in its appropriation, lest some confusion should arise in the mind of the young mathematician when he sees in Trigonometrical works, of the last age, a use different from our own of the word supplemental.

It may be remarked that the choice of the word for that purpose was not happy; for though it was so far a defining property as to give the *species* of the triangle, it did not give its *position*; an element quite as important, in many investigations, as the species itself, and, indeed, that upon which several of its valuable properties depend.

Again, for the angles, we have the angles

A B C of the fundamental triangle ABC

A B, C, those of BA'C

A,, B C,, those of CB'A

A,,, B,,, C those of AC'B.

The values of these are

$$\begin{array}{l|l} b_i = \pi - b; c_i = \pi - c & B_i = \pi - B; C_i = \pi - C \\ a_{ii} = \pi - a; c_{ii} = \pi - c & A_{ii} = \pi - A; C_{ii} = \pi - C \\ a_{iii} = \pi - a; b_{iii} = \pi - b & A_{iii} = \pi - A; B_{iii} = \pi - B \end{array}$$

Again, for the radii of the circles inscribed in the four triangles, taken in the aforesaid order, we write r, r_i, r_{ii}, r_{iii} : whilst, for the radii of the circumscribed circles, we put R, R_i, R_{ii}, R_{iii} , respectively. The unaccented letters referring to the circles of the fundamental triangles.

These triangles possess many beautiful properties when considered in their mutual association, which render them worthy of greater attention than has yet been bestowed upon them. Indeed, till very recently, their existence has scarcely been alluded to by writers on spherical subjects, and even to the present day, not more than three of their propositions have, we believe, been published.

2. Let O be the centre of the circle inscribed in the fundamental triangle, and G, H, K, its points of contact with the sides, join AO, BO, CO, and draw the radii to the points of contact. Then the tangents from A to the circle are equal; that is, $AK = AH$; in like manner $BK = BG$, and $CG = CH$.

Put

$$AK = AH = \alpha$$

$$BK = BG = \beta$$

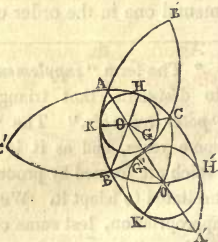
$$CG = CH = \gamma$$

From which

$$a = \beta + \gamma$$

$$b = \alpha + \gamma$$

$$c = \alpha + \beta$$



Whence

$$\left. \begin{aligned} a + \beta + \gamma &= \frac{a + b + c}{2} = s \\ a &= \frac{-a + b + c}{2} = \overline{s - a} \\ \beta &= \frac{a - b + c}{2} = \overline{s - b} \\ \gamma &= \frac{a + b - c}{2} = \overline{s - c} \end{aligned} \right\} \dots (1).$$

Again, in the right-angled triangle BOG, we have

$$\tan. OG = \sin. BG \tan. OBG,$$

that is,

$$\tan. r = \sin. \beta \tan. \frac{1}{2} B;$$

or by (1) just given and (3), upon page 77, applied to B, we have

$$\begin{aligned} \tan. r &= \sin. \overline{s - b} \left\{ \frac{\sin. \overline{s - a} \sin. \overline{s - c}}{\sin. s \sin. \overline{s - b}} \right\}^{\frac{1}{2}} \\ &= \frac{\sqrt{\sin. s \sin. \overline{s - a} \sin. \overline{s - b} \sin. \overline{s - c}}}{\sin. s} \dots (2). \end{aligned}$$

Again, in the supplemental triangle BA'C, denoting the quantities BB', CH', and AK', by $\beta_1, \gamma_1, \alpha_1$, we shall have

$$a_1 = \beta_1 + \gamma_1,$$

$$b_1 = \alpha_1 + \gamma_1,$$

$$c_1 = \alpha_1 + \beta_1,$$

and hence, as before,

$$\begin{aligned} s_1 = \alpha_1 + \beta_1 + \gamma_1 &= \frac{a_1 + b_1 + c_1}{2} = \frac{a + \pi - \overline{b} + \pi - c}{2} = \\ &= \pi - \frac{-a + b + c}{2} = \pi - \overline{s - a} \end{aligned}$$

$$s, - a, = \alpha, = \frac{-a, + b, + c,}{2} = \pi - \frac{-a + \overline{\pi - b} + \overline{\pi - c}}{2} =$$

$$\frac{a + b + c}{2} = s$$

$$\overline{s, - b,} = \beta, = \frac{a, - b, + c,}{2} = \frac{a - \overline{\pi - b} + \overline{\pi - c}}{2} =$$

$$\frac{a + b - c}{2} = \overline{s - c}$$

$$\overline{s, - c,} = \gamma, = \frac{a, + b, - c,}{2} = \frac{a + \overline{\pi - b} - \overline{\pi - c}}{2} =$$

$$\frac{a - b + c}{2} = \overline{s - b}.$$

Also

$$B, = \pi - B, \text{ and } \tan. \frac{1}{2} B, = \cot. \frac{1}{2} B.$$

Hence, in the right-angled triangle BO'G', we have

$$\tan. O'G' = \sin. BG \tan. O'BG',$$

that is,

$$\tan. r, = \sin. \beta, \tan. \frac{1}{2} B,$$

$$= \sin. \overline{s - c} \left\{ \frac{\sin. s \sin. \overline{s - b}}{\sin. s - a \sin. \overline{s - c}} \right\}^{\frac{1}{2}}$$

$$= \sqrt{\frac{\sin. s \sin. \overline{s - a} \sin. \overline{s - b} \sin. \overline{s - c}}{\sin. s - a}}$$

In exactly the same way we find the other associated inscribed radii, and the whole tabulated gives

$$\left. \begin{aligned} \tan. r &= \sqrt{\frac{\sin. s \sin. s - a \sin. s - b \sin. s - c}{\sin. s}} \\ \tan. r' &= \sqrt{\frac{\sin. s \sin. s - a \sin. s - b \sin. s - c}{\sin. s - a}} \\ \tan. r'' &= \sqrt{\frac{\sin. s \sin. s - a \sin. s - b \sin. s - c}{\sin. s - b}} \\ \tan. r''' &= \sqrt{\frac{\sin. s \sin. s - a \sin. s - b \sin. s - c}{\sin. s - c}} \end{aligned} \right\} \dots (3).$$

These formulæ were first given by *Professor Lowry* (1829), *Leybourn's Repository*, vol. v. p. 3.

Multiply these together, then we obtain

$$\tan. r \tan. r' \tan. r'' \tan. r''' = \sin. s \sin. s - a \sin. s - b \sin. s - c \dots (4).$$

Divide (4) by the squares of each of the equations in art. (3), the first side by the first side, and the second by the second: then

$$\left. \begin{aligned} \sin.^2 s &= \cot. r \tan. r' \tan. r'' \tan. r''' \\ \sin.^2 s - a &= \tan. r \cot. r' \tan. r'' \tan. r''' \\ \sin.^2 s - b &= \tan. r \tan. r' \cot. r'' \tan. r''' \\ \sin.^2 s - c &= \tan. r \tan. r' \tan. r'' \cot. r''' \end{aligned} \right\} \dots (5);$$

which remarkable formulæ are due to Mr. Lowry (1819), vide *Repository*, *ub. sup.*

Again, by multiplication of the terms in (3), we have

$$\begin{aligned} \tan. r \tan. r' + \tan. r'' \tan. r''' &= \sin. s - b \sin. s - c + \sin. s \sin. s - a \\ &= \sin. \frac{a + b - c}{2} \sin. \frac{a - b - c}{2} + \\ &\quad \sin. \frac{b + c + a}{2} \sin. \frac{b + c - a}{2} \end{aligned}$$

$$\begin{aligned}
&= \sin.^2 \frac{a}{2} - \sin.^2 \frac{b-c}{2} + \sin.^2 \frac{b+c}{2} - \sin.^2 \frac{a}{2} \\
&= \sin. b \sin. c.
\end{aligned}$$

Taking also each of the other corresponding combinations, we obtain in all the three following equations,

$$\left. \begin{aligned}
\tan. r \tan. r_i + \tan. r_{ii} \tan. r_{iii} &= \sin. b \sin. c \\
\tan. r \tan. r_{ii} + \tan. r_i \tan. r_{iii} &= \sin. a \sin. c \\
\tan. r \tan. r_{iii} + \tan. r_i \tan. r_{ii} &= \sin. a \sin. b
\end{aligned} \right\} \dots (6).$$

Or, by addition, we have at once the following theorem.

$$\begin{aligned}
&\tan. r \tan. r_i + \tan. r \tan. r_{ii} + \tan. r \tan. r_{iii} + \tan. r_i \tan. r_{ii} + \\
&\quad \tan. r_i \tan. r_{iii} + \tan. r_{ii} \tan. r_{iii} \\
&= \sin. a \sin. b + \sin. a \sin. c + \sin. b \sin. c \dots (7).
\end{aligned}$$

That is, in words, *the sum of the binary products of the tangents of the four inscribed radii are equal to the sum of the binary products of the sines of the sides.*

We may notice one beautiful theorem more, which is due to Mr. Lowry, *ubi. supra.* It is

$$\begin{aligned}
&\tan. r_i \tan. r_{ii} + \tan. r_i \tan. r_{iii} + \tan. r_{ii} \tan. r_{iii} = \\
&\sin. s \{ \sin. s - a + \sin. s - b + \sin. s - c \} \dots (8).
\end{aligned}$$

For the several consequences of these theorems, and a continuation of the inquiry, we must refer to the number xxiv. of Leybourn's Repository, now in the press, where expressions for the various trigonometrical functions of the sides and angles of the triangle, will be given in terms of the inscribed radii.

3. We now proceed to consider the *circumscribed* radii of the associated triangles. We shall immediately find these in terms of the angles, as we did those of the inscribed in terms of the sides.

Let Q be the centre of the circumscribing circle of the fundamental triangle, and draw the perpendiculars QM, QN, QP. Then M, N, P, bisect the sides *a*, *b*, *c*, respectively, and the several triangles BQC, CQA, AQB, are isosceles. Let the angles made by the radii QB, QC,

with the side a be denoted by α ; those with b , by β , and those with c by γ .*

Then

$$A = \beta + \gamma$$

$$B = \alpha + \gamma$$

$$C = \alpha + \beta.$$

From which we have

$$\left. \begin{aligned} \alpha + \beta + \gamma &= \frac{A + B + C}{2} = S \\ \alpha &= \frac{-A + B + C}{2} = S - A \\ \beta &= \frac{A - B + C}{2} = S - B \\ \gamma &= \frac{A + B - C}{2} = S - C \end{aligned} \right\} \dots (9).$$



But, by right-angled triangles BMQ, we have

$$\cot. QB = \cos. QBM \cot. BM,$$

or

$$\cot. R = \cos. \alpha \cot. \frac{1}{2} \alpha;$$

or, by (9) and (p. 80), we have at once

$$\cot. R = \sqrt{\frac{-\cos. S \cos. S - A \cos. S - B \cos. S - C}{-\cos. S}}.$$

Proceeding with respect to the triangle $BA'C$, in a manner analogous to that employed in obtaining the three last equations of (3), using the values of α, β, γ , just given in (9), we shall have the following tablet of values,

* These quantities being merely introduced as subsidiary ones, to be replaced in all general formulæ by their values in terms of A, B, C , we have not transgressed our general rule in employing them to designate two different sets of quantities in this place as in art. (2) which belong finally to the inquiry.

$$\left. \begin{aligned} \cot. R &= \sqrt{\frac{-\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C}}{-\cos. S}} \\ \cot. R_1 &= \sqrt{\frac{-\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C}}{\cos. \overline{S-A}}} \\ \cot. R_2 &= \sqrt{\frac{-\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C}}{\cos. \overline{S-B}}} \\ \cot. R_3 &= \sqrt{\frac{-\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C}}{\cos. \overline{S-C}}} \end{aligned} \right\} \dots (10);$$

which, with the following beautiful theorem, analogous to Lowry's, at p. 225, (obtained by multiplying these together) is due to Dr. Lardner, (1826). Trig. p. 153.

$$\cot. R \cot. R_1 \cot. R_2 \cot. R_3 = -\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C} \dots (11).$$

Divide (11) by the squares of each of the equations in (10), and we have

$$\left. \begin{aligned} \cos.^2 S &= \tan. R \cot. R_1 \cot. R_2 \cot. R_3 \\ \cos.^2 \overline{S-A} &= \cot. R \tan. R_1 \cot. R_2 \cot. R_3 \\ \cos.^2 \overline{S-B} &= \cot. R \cot. R_1 \tan. R_2 \cot. R_3 \\ \cos.^2 \overline{S-C} &= \cot. R \cot. R_1 \cot. R_2 \tan. R_3 \end{aligned} \right\} \dots (12).$$

These elegant theorems, which are here published for the first time, were discovered by my learned friend, the *Rev. H. F. C. Logan*, Professor of Mathematics in the Catholic College of Prior Park. The first of them is a remarkable expression for the *spherical excess in terms of the four circumscribed radii*. The spherical excess in terms of the *inscribed radii* may be seen in the Repository before alluded to; and some theorems connected with the same function of the triangle will be given in a future page of this supplement.

By combining (10) in the same way as (5) was combined to obtain (6), we shall

$$\left. \begin{aligned} \cot. R \cot. R_1 + \cot. R_2 \cot. R_3 &= \sin. B \sin. C \\ \cot. R \cot. R_2 + \cot. R_1 \cot. R_3 &= \sin. A \sin. C \\ \cot. R \cot. R_3 + \cot. R_1 \cot. R_2 &= \sin. A \sin. B \end{aligned} \right\} \dots (13).$$

Hence, by addition, we get

$$\begin{aligned} \cot. R \cot. R_1 + \cot. R \cot. R_2 + \cot. R \cot. R_3 + \cot. R_1 \cot. R_2 + \\ \cot. R_1 \cot. R_3 + \cot. R_2 \cot. R_3 \\ = \sin. A \sin. B + \sin. B \sin. C + \sin. C \sin. A \dots (14). \end{aligned}$$

In the Repository (xxiv) will also be found expressions for the trigonometrical functions of the elements of the triangle, in terms of R , R_1 , R_2 , R_3 ; and we may here remark that by means of a theorem to be given at page 235 of this treatise, the expressions (10, 14, incl. and all of the same class) may be derived, by inspection, from those given in terms of sides and inscribed radii. It is by means of a property of the polar triangle.

We shall, however, before proceeding to the theory of polar triangles, point the student's attention to two interesting propositions, the analytical expressions for which we have passed by without particular notice. We allude to the values of $\tan. r$, and of $\cot. R$, at pages 223, 227, and which are

$$\left. \begin{aligned} \tan. r &= \sin. \beta \tan. \frac{1}{2} B = \sin. \frac{1}{2} (a + c - b) \tan. \frac{1}{2} B \\ \cot. R &= \cos. a \cot. \frac{1}{2} a = \cos. \frac{1}{2} (B + C - A) \cot. \frac{1}{2} a \end{aligned} \right\}$$

1. From the first of these we infer that if B and $a + c - b$ are constant, r will be constant; that is to say, *in any spherical triangle if the vertical angle (B) be constant, as also the difference between the base and sum of the other two sides, the radius and centre of the inscribed circle will continue fixed.*

2. From the second we infer that, *in any spherical triangle if the base (a) be constant, as also the difference between the vertical angle, and the sum of the other two, the radius and centre of the circumscribed circle will be fixed; that is, the locus of the vertex will be a circle.*

3. This last property suggests a remarkably simple method of demonstrating the beautiful theorem of *Lexell* which is this, viz. that

if the base and area of the spherical triangle be constant, the locus of the vertex will be a circle.

For, referring to the figure at page 227, let BC be the constant base, and ABC any one of the triangles. Produce the sides to meet in A', and call the angles at B and C *below* the base B' and C'.

For the area of the triangle ABC we have the expressions

$$A + B + C - \pi = \text{constant.}$$

But

$$A = A', B = \pi - B', C = \pi - C';$$

hence, by substitution,

$$A' - (B' + C') + \pi = \text{const.} \therefore B' + C' - A' = \text{const.}$$

and, therefore, as the base BC or a' is also constant, it follows from the theorem just demonstrated that the locus of the vertex A' is the circle A'BC, and, consequently, the locus of A, which is the antipodes of A', must be an equal circle. We ought to remark here that this demonstration is the same in substance as that given by Mr. Lowry in Leybourn's Repository, vol. 1.*

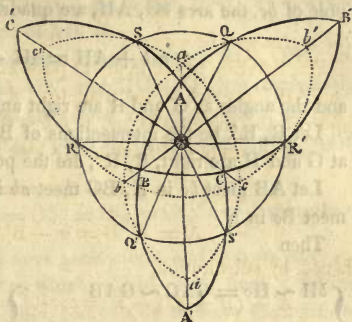
Polar Triangles.

We have already seen, (p. 70,) that if from the three angular points of a spherical triangle ABC we describe three great circles, they will form an associated system of triangles, *one* of which also has a remarkable relation to the triangle ABC: but it has not, so far as we know, been noticed, that if we complete the associated system, whose fundamental is ABC, then the two sets of associated triangles thus produced, will be separable into four pairs, (one of each system forming a pair,) which will be related to one another precisely in the same way as the above-named pair are related. Thus, if in the annexed figure ABC, be the fundamental triangle, abc its polar; then four pairs of polars will simultaneously be produced, viz.

$$(1) \frac{ABC}{abc} \mid (2) \frac{AB'C}{ab'c} \mid (3) \frac{A'BC}{a'bc} \mid (4) \frac{ABC'}{abc'}.$$

* In the Edinburgh Transactions the author of this supplement has investigated Lexell's theorem by a novel analysis,—*the geometry of spherical coordinates*, which determines in a direct manner the equation of the locus, and then shews its identity with the previously found equation of a circle.

Now the first pair ABC, abc are by hypothesis *polars*.* Hence B is the pole of ca , that is of ac' ; A is the pole of bc , that is of bc' ; and since C is the pole of ab , therefore C' is the other pole of ab . The points A, B, C' are the poles of the sides, therefore, of the triangle abc' . Whence, also, it follows by the reciprocity of the polar system, that a, b, c' are the poles of ABC' .



In the same way it is shown that ACB' and acb' form a polar system, and that $BA'C$, $ba'c$, form another. Hence it follows that if one triangle of an associated system be polar to one of another associated system, then each of the triangles of one system is polar to one triangle of the other associated system.

This is not the only curious property of the figure before us: and we shall put down a small selection from those we are in possession of, not doubting that on many accounts they will be interesting to those geometers who indulge in trigonometrical speculations. An addition to these will appear in the 25th number of the Mathematical Repository.

Draw the great circle aA (next figure), and produce it to meet BC , bc in G and H . Then, because a is the pole of BC , and that A is the

* When we speak of parallel lines, without specifying which is taken as the line of reference, and knowing that the second is related to the first in the same way that the first is to the second, we simply denominate them parallels. The same practice also holds in speaking of two mutually supplemental angles. But when we previously fix upon one line or one angle, we say the parallel, or the supplement in the singular number to express the other line or angle. Just so in respect to the two triangles which constitute a polar system, when we speak of both without assigning the referee, we call them polars, as a common epithet; but when we have fixed upon one already, as that to which the other is referred, we call it the *primary*, and the other the *polar triangle*.

pole of bc , the arcs aG , AH , are quadrants. Hence,

$$aG + AH = aH + AG = \pi,$$

and the angles at G and H are right angles.

Let R , R' , be the intersections of BC , bc ; then, because the angles at G and H are right, R , R' , are the poles of aA .

Let AB meet bc in K , BC meet ab in L , AC meet bc in M , and ac meet Bc in N .

Then

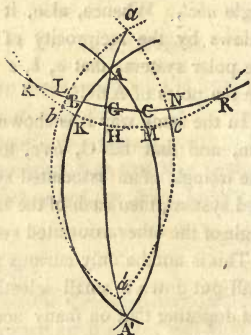
$$\left\{ \begin{array}{l} bH \sim Hc = GAC \sim GAB \\ bH + GAB = \frac{\pi}{2} = Hc + GAC. \end{array} \right\}$$

For

$$bM = bH + HM = bH + HAM = bH + GAC$$

In like manner,

$$cK = cH + HK = cH + HAK = cH + GAB.$$



Also $bM = \frac{\pi}{2} = cK$, since b and c are the poles AM and CK .

Hence the propositions as stated are true.

These are due to *Professor Lowry*, and were given by him, in 1800, in *Leybourn's Repos.* old series, No. 1. p. 44, 5. They have not, that we are aware, been noticed in any other place.

Returning now to our original figure, let the three arcs Aa , Bb , Cc , be drawn; these will pass through the same point O , because as has been just shown, they are perpendicular to the three sides BC , AC , AB , (vide p. 219). Also, because aA is perpendicular to bc , it will pass through the opposite pole a' . In like manner it will pass through A' . Or the points $aAOa'A'$ are in the same great circle, whose poles are R and R' , the intersections of BC , bc .

In like manner $bBOb'B'$ are in one great circle, whose poles are S , S' ,

the intersections of AC, ac ; $cCOc'C'$ are in one great circle, whose poles are Q, Q' , the intersections of AB, ab .

Again, because R, R' , are the poles of aa' , the arcs RO, OR' , are quadrants. In like manner SO, OS' , and QO, OQ' , are respectively quadrants; and as the quadrants are drawn from the poles of aa', bb', cc' , respectively, R, O, R' ; S, O, S' , and Q, O, Q' , are respectively in the same great circles.

Also since $OR = OR' = OS = OS' = OQ = OQ' = \frac{\pi}{2}$, the points

R, R', S, S', A, Q' , are in the same great circle, and O is its pole.

The complexity of the figure which would result from a further detail of these interesting researches, compels us to leave them for the present to the industry and ingenuity of the student. They are exceedingly easy, and furnish an excellent exercise in spherical investigations: and we therefore hope he will give it a proper degree of consideration. We proceed to view the subject algebraically.

Let abc, ABC , be the sides and angles respectively of a spherical triangle; R, r the radii of the circumscribing and inscribed circles; s, S , the semi sums of the sides and angles respectively; also let $a'b'c'; A'B'C'$; R', r' ; S', s' , be the same things in the triangle which is polar to abc .

Put, as at page 199,

$$n^2 = \sin. s \sin. \overline{s-a} \sin. \overline{s-b} \sin. \overline{s-c}$$

$$n'^2 = \sin. s' \sin. \overline{s'-a'} \sin. \overline{s'-b'} \sin. \overline{s'-c'}$$

$$N^2 = -\cos. S \cos. \overline{S-A} \cos. \overline{S-B} \cos. \overline{S-C}$$

$$N'^2 = -\cos. S' \cos. \overline{S'-A'} \cos. \overline{S'-B'} \cos. \overline{S'-C'}$$

Then we propose to prove that $n^2 = N'^2$, and $n'^2 = N^2$.

For

$$s = \frac{1}{2} (a + b + c)$$

$$s - a = \frac{1}{2} (-a + b + c)$$

$$s - b = \frac{1}{2} (a - b + c)$$

$$s - c = \frac{1}{2} (a + b - c);$$

and, in the polar triangle, we have immediately

$$S' = \frac{3\pi}{2} - \frac{1}{2}(a + b + c) = \frac{3\pi}{2} - s$$

$$S' - A' = \frac{\pi}{2} - \frac{1}{2}(-a + b + c) = \frac{\pi}{2} - \overline{s - a}$$

$$S' - B' = \frac{\pi}{2} - \frac{1}{2}(a - b + c) = \frac{\pi}{2} - \overline{s - b}$$

$$S' - C' = \frac{\pi}{2} - \frac{1}{2}(a + b - c) = \frac{\pi}{2} - \overline{s - c}.$$

Hence,

$$-\cos. S' = -\cos. \left(\frac{3\pi}{2} - s\right) = \sin. s$$

$$\cos. S' - A' = \cos. \left(\frac{\pi}{2} - \overline{s - a}\right) = \sin. \overline{s - a}$$

$$\cos. \overline{S' - B'} = \cos. \left(\frac{\pi}{2} - \overline{s - b}\right) = \sin. \overline{s - b}$$

$$\cos. \overline{S' - C'} = \cos. \left(\frac{\pi}{2} - \overline{s - c}\right) = \sin. \overline{s - c}.$$

Whence, by multiplication of the extreme vertical columns we have the quantities designated by N'^2 , and n^2 ; or

$$N'^2 = n^2.$$

In exactly the same way, we find that

$$N^2 = n'^2.$$

That is,

$$N' = n, \text{ and } N = n'.^* \quad . \quad . \quad . \quad .$$

* These properties might have been more simply obtained, thus: by (18) and (21), page 201.

$$\tan. \frac{1}{2}A' \tan. \frac{1}{2}B' \tan. \frac{1}{2}C' = \frac{n'}{\sin.^2 S'}$$

$$\cot. \frac{1}{2}a \cot. \frac{1}{2}b \cot. \frac{1}{2}c = \frac{N}{\cos.^2 S}$$

Again, by Lowry's theorem, p. 225,

$$\tan. r = \frac{n}{\sin. s}, \tan. r' = \frac{n'}{\sin. s'};$$

and, therefore, by (17, p. 201),

$$\tan. r' = \frac{N'}{2 \cos. \frac{1}{2} A' \cos. \frac{1}{2} B' \cos. \frac{1}{2} C'} = \frac{N'}{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}.$$

Also, (10) p. 228, and (19) p. 201,

$$\tan. R = -\frac{\cos. S}{N} = \frac{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{n}$$

$$\tan. R' = -\frac{\cos. S'}{N} = \frac{\sin. s}{N'}.$$

Whence, recollecting that $N = n$,

$$\frac{\tan. r}{\tan. r'} = \frac{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\sin. \frac{1}{2} (a + b + c)}.$$

and

$$\frac{\tan. R}{\tan. R'} = \frac{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\sin. \frac{1}{2} (a + b + c)}$$

$$\therefore \frac{\tan. r}{\tan. r'} = \frac{\tan. R}{\tan. R'}.$$

But we have a more useful result, as follows:

But

$$\frac{1}{2} (A' + a) = \frac{\pi}{2} \therefore \cot. \frac{1}{2} a = \tan. \frac{1}{2} A',$$

and hence the left sides are equal. Also, $\cos. ^2 S = \sin. ^2 s$, therefore

$$N = n'.$$

Had the mere proof of the property been our object, this would have been the preferable method: but the stages through which in the text we have passed are necessary in other enquiries, and we preferred therefore to give them in that form which, it may be remarked too, is the original one by which the property was obtained. See note E.

$$\tan. R \tan. r' = \frac{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{n} \cdot \frac{N'}{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}.$$

$$= 1, \text{ or}$$

$$\tan. R = \cot. r',$$

which is fulfilled by the relations

$$R + r' = \frac{\pi}{2}, \text{ and } R + r' = \frac{3\pi}{2}.$$

The former of these relations holds when the circles are referred to their nearest poles, and the latter when to their farther poles.

In like manner, we also have, in the corresponding cases,

$$R' + r = \frac{\pi}{2} \text{ and } R' + r = \frac{3\pi}{2}.$$

From these we get

$$R + r' = R' + r$$

$$R - R' = r - r'$$

$$R - r = R' - r'$$

Thus the remarkable relation between the polar triangles is continued even amongst the radii of the inscribed and circumscribing circles, viz. *that the inscribed radius of one is the complement of the circumscribing radius of the other; or, taking diameters, the supplement of the other.*

We have seen that in the two associated systems, having one pair of mutually polar fundamental triangles, the two systems are, triangle for triangle mutually polar; and hence collecting the whole result into one table, we have

$$R + r = \frac{\pi}{2}$$

$$R + r' = \frac{\pi}{2}$$

$$R' + r' = \frac{\pi}{2}$$

$$R_1 + r'_1 = \frac{\pi}{2}$$

$$R'' + r'' = \frac{\pi}{2}$$

$$R'' + r'' = \frac{\pi}{2}$$

$$R''' + r''' = \frac{\pi}{2}$$

$$R''' + r''' = \frac{\pi}{2}$$

$$R' + r = \frac{3\pi}{2}$$

$$R + r' = \frac{3\pi}{2}$$

$$R' + r_1 = \frac{3\pi}{2}$$

$$R_1 + r'_1 = \frac{3\pi}{2}$$

$$R'' + r'' = \frac{3\pi}{2}$$

$$R'' + r'' = \frac{3\pi}{2}$$

$$R''' + r''' = \frac{3\pi}{2}$$

$$R''' + r''' = \frac{3\pi}{2}$$

Adding all these together, we have

$$r + r_1 + r'' + r''' + r' + r'_1 + r'_2 + r'_3 + R + R_1 + R'' + R''' + R' + R'_1 + R'_2 + R'_3 = 4\pi \text{ (or } 12\pi \text{)}.$$

That is, the sum of all the sixteen radii of the primary and polar associated system is equal to the surface of the hemisphere if the first system of values is taken, or to three times that sum if the second system be taken.

The consideration of the latter system of values I owe to the Rev. Professor Logan; the former with its results is my own independent discovery, and was the origin of my researches on the subject of polar triangles, and of the associated triangles too.

Another property also comes immediately from this: viz. that the product of the tangents of the sixteen radii is equal to unity. Which is seen at once by $\tan. r. \tan. R' = 1$, &c: or it may again be put

$$\tan. r. \tan. r_1. \tan. r'' . \tan. r''' . \tan. r' . \tan. r'_1 . \tan. r'_2 . \tan. r'_3 =$$

$$\cot. R. \cot. R_1. \cot. R'' . \cot. R''' . \cot. R' . \cot. R'_1 . \cot. R'_2 . \cot. R'_3;$$

or, still differently,

$$\frac{\tan. r \tan. r' \tan. r'' \tan. r'''}{\cot. R \cot. R' \cot. R'' \cot. R'''} = \frac{\cot. r' \cot. r'' \cot. r''' \cot. r'''}{\tan. R' \tan. R'' \tan. R''' \tan. R''''}$$

Another neat relation may be put down here; we have already seen that

$$\frac{\tan. R}{\tan. R'} = \frac{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\sin. \frac{1}{2} (a + b + c)};$$

and by interchanging the primary and secondary polar triangles, still retaining the accent upon the same letters to distinguish them as the sides of the same triangle as before, we have

$$\frac{\tan. R'}{\tan. R} = \frac{2 \sin. \frac{1}{2} a' \sin. \frac{1}{2} b' \sin. \frac{1}{2} c'}{\sin. \frac{1}{2} (a' + b' + c')}.$$

Dividing these, we have

$$\frac{\tan. {}^2R}{\tan. {}^2R'} = \frac{\sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\sin. \frac{1}{2} a' \sin. \frac{1}{2} b' \sin. \frac{1}{2} c'} \cdot \frac{\sin. \frac{1}{2} (a' + b' + c')}{\sin. \frac{1}{2} (a + b + c)}.$$

But $\frac{1}{2}a = \frac{\pi}{2} - \frac{1}{2}A$, &c., because the triangles are polar;

$$\begin{aligned} \therefore \frac{\tan. {}^2R}{\tan. {}^2R'} &= \frac{\sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C} \cdot \frac{-\cos. S}{\sin. S} \\ &= \frac{\sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c}{\sin. S} \cdot \frac{-\cos. S}{\cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C}. \end{aligned}$$

Multiplying the same equations, we get

$$4 \cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c = -\cos. S \sin. s;$$

or, in every spherical triangle we have

$$\frac{2 \cos. \frac{1}{2} A \cos. \frac{1}{2} B \cos. \frac{1}{2} C}{\cos. S} = \frac{\sin. s}{2 \sin. \frac{1}{2} a \sin. \frac{1}{2} b \sin. \frac{1}{2} c},$$

which, by the bye, is also an immediate consequence of the relations (17), (19), at page 201.

Innumerable other interesting results may be obtained, with equal

facility, by means of the property of polar radii given above: but the limits of a work, like the present, prevent our enlarging upon them here.

We may, however, refer for some of them to the Mathematical Repository, No. xxiv.

CHAPTER III.

SOME ADDITIONAL INQUIRIES RESPECTING THE SPHERICAL EXCESS.

We shall now devote a short chapter to some miscellaneous inquiries respecting the Spherical Excess, in continuation of what has been already done in Chapter II. Part IV. All the usual formulæ for the spherical excess have there been amply discussed; but there are still certain other combinations of data which have not yet been considered: these are, 1st. Two angles and the interjacent side; 2d. Two angles and a side opposite to one of them: and lastly, Two sides and an angle opposite to one of them. Expressions for the spherical excess in each of these cases may be readily deduced. In the last two, however, the formulæ which I have obtained are neither sufficiently symmetrical nor sufficiently simple to render them deserving of much notice, either for analytical beauty or for practical convenience; they involve, however, but one radical. The formula for the first of the above cases I investigate as follows.

To determine the Spherical Excess when two angles and the interjacent side are given.

Here we have

$$\frac{E}{2} = \frac{A + B + C - 180^\circ}{2},$$

and therefore

$$\cos. \frac{E}{2} = \cos. \frac{A+B}{2} \sin. \frac{C}{2} + \sin. \frac{A+B}{2} \cos. \frac{C}{2},$$

$$\sin. \frac{E}{2} = \cos. \frac{A+B}{2} \cos. \frac{C}{2} - \sin. \frac{A+B}{2} \sin. \frac{C}{2}.$$

But $\sin.^2 \frac{C}{2}$ and $\cos.^2 \frac{C}{2}$ may take either of the following forms:

$$\begin{aligned} \sin.^2 \frac{C}{2} &= \frac{1 - \cos. C}{2} = \frac{1 + \cos. A \cos. B - \sin. A \sin. B \cos. c}{2} \\ &= \cos.^2 \frac{A}{2} \cos.^2 \frac{B}{2} - 2 \cos. \frac{A}{2} \sin. \frac{A}{2} \cos. \frac{B}{2} \sin. \frac{B}{2} \cos. c \\ &\quad + \sin.^2 \frac{A}{2} \sin.^2 \frac{B}{2} \\ &= \cos.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \cos.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2} \\ \cos.^2 \frac{C}{2} &= \frac{1 + \cos. C}{2} = \frac{1 - \cos. A \cos. B + \sin. A \sin. B \cos. c}{2} \\ &= \cos.^2 \frac{A}{2} \sin.^2 \frac{B}{2} - 2 \cos. \frac{A}{2} \sin. \frac{A}{2} \cos. \frac{B}{2} \sin. \frac{B}{2} \cos. c \\ &\quad + \sin.^2 \frac{A}{2} \cos.^2 \frac{B}{2} \\ &= \sin.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \sin.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2}. \end{aligned}$$

Thus we shall have a choice of three forms, to suit the specific purpose we have in view. The last is the preferable on the ground of *algebraical symmetry*. Substituting these, we have

$$\begin{aligned} \cos. \frac{E}{2} &= \cos. \frac{A+B}{2} \left\{ \cos.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \cos.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2} \right\}^{\frac{1}{2}} \\ &\quad + \sin. \frac{A+B}{2} \left\{ \sin.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \sin.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2} \right\}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \sin. \frac{E}{2} = \cos. \frac{A+B}{2} \left\{ \sin.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \sin.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2} \right\}^{\frac{1}{2}} \\ + \sin. \frac{A+B}{2} \left\{ \cos.^2 \frac{A-B}{2} \sin.^2 \frac{c}{2} + \cos.^2 \frac{A+B}{2} \cos.^2 \frac{c}{2} \right\}^{\frac{1}{2}} \end{aligned}$$

To determine the excess, when the three sides are given.

This case has been already discussed, but the following investigation may not be unacceptable.

By (46, 4,)

$$\begin{aligned} \tan. \frac{E}{4} &= \frac{2 \sin. \frac{A+B+C-\pi}{4} \cos. \frac{A+B-C+\pi}{4}}{2 \cos. \frac{A+B+C-\pi}{4} \cos. \frac{A+B-C+\pi}{4}} \\ &= \frac{\sin. \frac{A+B}{2} - \sin. \left(\frac{\pi}{2} - \frac{C}{2} \right)}{\cos. \frac{A+B}{2} + \cos. \left(\frac{\pi}{2} - \frac{C}{2} \right)} \\ &= \frac{\sin. \frac{A+B}{2} - \cos. \frac{C}{2}}{\cos. \frac{A+B}{2} + \sin. \frac{C}{2}}, \end{aligned}$$

or by (art. 86)

$$\begin{aligned} &= \frac{\cos. \frac{a-b}{2} - \cos. \frac{c}{2}}{\cos. \frac{a+b}{2} + \cos. \frac{c}{2}} \cot. \frac{C}{2}, \\ &= \frac{\sin. \frac{s-a}{2} \sin. \frac{s-b}{2}}{\cos. \frac{s}{2} \cos. \frac{s-c}{2}} \cot. \frac{C}{2} \dots (a). \end{aligned}$$

But

$$\cot. \frac{C}{2} = \sqrt{\frac{\sin. s \sin. s - c}{\sin. s - a \sin. s - b}}$$

$$= \sqrt{\frac{\sin. \frac{s}{2} \cos. \frac{s}{2} \sin. \frac{s-c}{2} \cos. \frac{s-c}{2}}{\sin. \frac{s-a}{2} \cos. \frac{s-a}{2} \sin. \frac{s-b}{2} \cos. \frac{s-b}{2}}} \dots (b).$$

Inserting (b) in (a), we have, after slight reductions,

$$\tan. \frac{E}{4} = \sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2}}$$

which is the remarkable formula of *Lhuillier*.*

Applying this to the polar triangles, some interesting results may be obtained as follows:

Denoting by S, S_a, S_m , the semi-sums of the sides of the supplementary triangles; by a, b , and c , the sides of $BA'C$; a_m, b, b_m , the sides of $AB'C$; and by a_m, b_m, c , the sides of $AC'B$. Then (see p. 223)

$$s = \frac{a + \pi - b + \pi - c}{2} = \frac{a - b - c}{2} + \pi =$$

$$\pi - \frac{-a + b + c}{2} = \pi - s - a$$

$$s - a = \frac{a - b - c}{2} + \pi - a = \pi - \frac{a + b + c}{2} = \pi - s$$

* The excess has also been obtained, by means of the modern analysis, by *Euler* in the Memoirs of the Royal Academy of Berlin, vol. ix. p. 256, and by *Tedénat*, in Gergonne's Annals of Mathematics, vol. vi. p. 48.

$$s - b = \frac{a - b - c}{2} + \pi - \pi - b = \frac{a + b - c}{2} = s - c$$

$$s - c = \frac{a - b - c}{2} + \pi - \pi - c = \frac{a - b + c}{2} = s - b.$$

Hence if E, E', E'' , denote the excesses of $BA'C, AB'C, BC'A$, we have by Lhuillier's theorem,

$$\tan. \frac{E}{4} = \sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2}},$$

that is

$$\left. \begin{aligned} \tan. \frac{E}{4} &= \sqrt{\cot. \frac{s}{2} \cot. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2}} \\ \tan. \frac{E'}{4} &= \sqrt{\cot. \frac{s}{2} \tan. \frac{s-a}{2} \cot. \frac{s-b}{2} \tan. \frac{s-c}{2}} \\ \tan. \frac{E''}{4} &= \sqrt{\cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \cot. \frac{s-c}{2}} \\ \tan. \frac{E'''}{4} &= \sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2}} \end{aligned} \right\} \dots (1).$$

in which Lhuillier's theorem is applied to each of the triangles in succession.

If we multiply these together, we find

$$\tan. \frac{E}{4} \tan. \frac{E'}{4} \tan. \frac{E''}{4} \tan. \frac{E'''}{4} = \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \dots (2)$$

Again, the angles of the triangle $BA'C$ are

$$A, = A, B, = \pi - B, \text{ and } C, = \pi - C.$$

Hence

Similarly

$$\left. \begin{aligned} \frac{E_1}{4} &= \frac{\pi + A - B - C}{4} \\ \frac{E_2}{4} &= \frac{\pi - A + B - C}{4} \\ \frac{E_3}{4} &= \frac{\pi - A - B + C}{4} \\ \frac{E}{4} &= \frac{-\pi + A + B + C}{4} \end{aligned} \right\} \dots (3).$$

Also

$$\frac{\pi + A - B - C}{4} + \frac{\pi - A + B + C}{4} = \frac{\pi}{2},$$

whence

$$\tan. \frac{\pi - A + B + C}{4} = \cot. \frac{E_1}{4} =$$

$$\sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \cot. \frac{s-b}{2} \cot. \frac{s-c}{2}};$$

and, by similar processes with the other triangles, we get the following table;

$$\left. \begin{aligned} \tan. \frac{-A + B + C + \pi}{4} &= \sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \cot. \frac{s-b}{2} \cot. \frac{s-c}{2}} \\ \tan. \frac{A - B + C + \pi}{4} &= \sqrt{\tan. \frac{s}{2} \cot. \frac{s-a}{2} \tan. \frac{s-b}{2} \cot. \frac{s-c}{2}} \\ \tan. \frac{A + B - C + \pi}{4} &= \sqrt{\tan. \frac{s}{2} \cot. \frac{s-a}{2} \cot. \frac{s-b}{2} \tan. \frac{s-c}{2}} \\ \tan. \frac{A + B + C - \pi}{4} &= \sqrt{\tan. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2}} \end{aligned} \right\} (4).$$

The last of which is the common form of the area of a triangle given by *Lhuillier*, applied to the fundamental triangle.

Multiply all these together, and we shall have

$$\tan.^2 \frac{a+b+c}{4} = \tan. \frac{A+B+C-\pi}{4} \cdot \tan. \frac{A+B-C+\pi}{4} \\ \tan. \frac{A-B+C+\pi}{4} \tan. \frac{-A+B+C+\pi}{4} \dots (5).$$

Also, giving to the terms of Lhuillier's theorem their unabbreviated values, we shall see a striking analogy in their general form between that and the one just obtained. For in

$$\tan.^2 \frac{A+B+C-\pi}{4} = \tan. \frac{a+b+c}{4} \tan. \frac{a+b-c}{4} \tan. \frac{a-b+c}{4} \\ \tan. \frac{-a+b+c}{4} \dots (6).$$

we see the only difference, as to general form, is that π enters into all the angular functions, and not into those of the sides.

Again, since the three last factors in the right hand member of equation (5) are $\cot. \frac{E'''}{4}$, $\cot. \frac{E''}{4}$, and $\cot. \frac{E'}{4}$; and the remaining factor

is $\tan. \frac{E}{4}$, we have (5) converted into

$$\tan.^2 \frac{a+b+c}{4} = \tan. \frac{E}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \dots (7).$$

By the principle of the symmetry of the triangles, and of their expressions, we at once infer from (7) that

$$\cot. \frac{E}{4} \tan. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} = \tan.^2 \frac{s_r}{2}.$$

But

$$s_r = \frac{a + \pi - b + \pi - c}{2} = \pi - \frac{-a + b + c}{2}$$

$$\therefore \tan. \frac{s_r}{2} = \cot. \frac{-a + b + c}{4} = \cot. \frac{s-a}{2}.$$

Applying the same principle of reduction to the other supplemental triangles, and collecting the results, we have

$$\left. \begin{aligned} \tan.^2 \frac{s}{2} &= \tan. \frac{E}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{s-a}{2} &= \cot. \frac{E}{4} \tan. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{s-b}{2} &= \cot. \frac{E}{4} \cot. \frac{E'}{4} \tan. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{s-c}{2} &= \cot. \frac{E}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \tan. \frac{E'''}{4} \end{aligned} \right\} \dots (8).$$

Let us resume equations (3), and multiply by 4, then we have

$$\left. \begin{aligned} A + B + C - \pi &= E \\ A - B - C + \pi &= E' \\ -A + B - C + \pi &= E'' \\ -A - B + C + \pi &= E''' \end{aligned} \right\} \dots (9).$$

add them, then

$$E + E' + E'' + E''' = 2\pi \dots (10).$$

Add the first of these to each of the others successively, then

$$\left. \begin{aligned} A &= \frac{E + E'}{2} \\ B &= \frac{E + E''}{2} \\ C &= \frac{E + E'''}{2} \end{aligned} \right\} \dots (11).$$

$$S = \frac{A + B + C}{2} = \frac{3E + E' + E'' + E'''}{4},$$

or, inserting the value of $E + E' + E'' + E'''$ from (10), it becomes

$$S = \frac{1\pi}{2} + \frac{E}{2};$$

and, in like manner, we have

$$\left. \begin{aligned} S - A &= \frac{E - E' + E'' + E'''}{4} = \frac{\pi}{2} - \frac{E'}{2} \\ S - B &= \frac{E + E' - E'' + E'''}{4} = \frac{\pi}{2} - \frac{E''}{2} \\ S - C &= \frac{E + E' + E'' - E'''}{4} = \frac{\pi}{2} - \frac{E'''}{2} \end{aligned} \right\} \dots (12).$$

$$\left. \begin{aligned} \cos. S &= \sin. \frac{E}{2} \\ \cos. \overline{S - A} &= \sin. \frac{E'}{2} \\ \cos. \overline{S - B} &= \sin. \frac{E''}{2} \\ \cos. \overline{S - C} &= \sin. \frac{E'''}{2} \end{aligned} \right\} \dots (13).$$

Inserting these values in the usual formula for finding a side, we get

$$\left. \begin{aligned} \cot.^2 \frac{1}{2} a &= \frac{\sin. \frac{E''}{2} \sin. \frac{E'''}{2}}{\sin. \frac{E}{2} \sin. \frac{E'}{2}} \\ \cot.^2 \frac{1}{2} b &= \frac{\sin. \frac{E'}{2} \sin. \frac{E'''}{2}}{\sin. \frac{E}{2} \sin. \frac{E''}{2}} \\ \cot.^2 \frac{1}{2} c &= \frac{\sin. \frac{E'}{2} \sin. \frac{E''}{2}}{\sin. \frac{E}{2} \sin. \frac{E'''}{2}} \end{aligned} \right\} \dots (14).$$

$$\left. \begin{aligned} \cot.^2 \frac{1}{2} a \cot.^2 \frac{1}{2} b \cot.^2 \frac{1}{2} c &= \frac{\sin. \frac{E''}{2} \sin. \frac{E'''}{2} \sin. \frac{E'''}{2}}{\sin. \frac{E}{2} \sin. \frac{E'}{2} \sin. \frac{E''}{2}} \\ &= \frac{\sin. \frac{E'}{2} \sin. \frac{E''}{2} \sin. \frac{E'''}{2}}{\sin.^3 \frac{E' + E'' + E'''}{2}} \end{aligned} \right\} \dots (15).$$

The sides and angles of the triangle are thus found, (the angles in 11,) in terms of the areas of the four triangles: and the equation of condition also which subsists among these four triangles is assigned in (10).

By (12) the values of the factors in N are found, and by (8) there is another trigonometrical function of the factors of n assigned. From this, those factors themselves may be assigned, but the process is troublesome and the result inelegant. We have obtained a simpler form, but even then neither the form nor the method is well suited to this place. The values of the inscribed and circumscribed radii in terms of the excesses will be discussed in the Repository, and we shall conclude this section with assigning the connexion between the polar systems of associated triangles, in respect to the areas.

The sides of the primary fundamental triangle being a, b, c , we have

$$* 2S' = 3\pi - (a + b + c), \text{ and hence } \frac{E'}{4}$$

$$\frac{E'}{4} = \frac{2\pi - (a + b + c)}{4} = \frac{\pi}{2} - \frac{a + b + c}{4}$$

and

$$\tan. \frac{E'}{4} = \cot. \frac{a + b + c}{4} = \cot. \frac{s}{2}.$$

Again, in the supplemental polar triangles, (that taken with respect to A for instance,) we have

$$2S'_s = (\pi - a) + (\pi - \overline{\pi - b}) + (\pi - \overline{\pi - c}) = \pi - a + b + c$$

$$\frac{E_s}{4} = \frac{\pi + (\pi - a + b + c)}{4} - \pi = \frac{-a + b + c}{4} = \frac{\overline{s - a}}{2}$$

and

$$\tan. \frac{E_s}{4} = \tan. \frac{s - a}{2}.$$

In performing the same changes upon $2S''$, and $2S'''$, we have the

* The accented letters $S' E'$, &c. denote quantities in the polar triangles which are denoted in the primary by S, E , &c.

values of $\frac{E'}{4}$ and $\frac{E''}{4}$; the whole form of which are collected together below.

$$\left. \begin{aligned} \tan. \frac{E'}{4} &= \cot. \frac{s}{2} \\ \tan. \frac{E'}{4} &= \tan. \frac{s-a}{2} \\ \tan. \frac{E''}{4} &= \tan. \frac{s-b}{2} \\ \tan. \frac{E''}{4} &= \tan. \frac{s-c}{2} \end{aligned} \right\} \dots (16).$$

Multiply these together; then

$$\tan. \frac{E'}{4} \tan. \frac{E'}{4} \tan. \frac{E''}{4} \tan. \frac{E''}{4} = \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \dots (16).$$

And, again, by comparison of (2) and (16), we find

$$\tan. \frac{E'}{4} \tan. \frac{E'}{4} \tan. \frac{E''}{4} \tan. \frac{E''}{4} = \tan. \frac{E}{4} \tan. \frac{E}{4} \tan. \frac{E''}{4} \tan. \frac{E''}{4} \dots (17).$$

By a comparison of the component equations of table (15) with those of table (8), we shall get the value of the area of any triangle in terms of the associated system which is polar to it. Thus,

$$\left. \begin{aligned} \cot.^2 \frac{E'}{4} &= \tan. \frac{E}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E'}{4} &= \cot. \frac{E}{4} \tan. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E''}{4} &= \cot. \frac{E}{4} \cot. \frac{E'}{4} \tan. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E''}{4} &= \cot. \frac{E}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \tan. \frac{E'''}{4} \end{aligned} \right\} \dots (18).$$

and conversely, by interchanging the system of reference in the polar

triangles we have

$$\left. \begin{aligned} \cot.^2 \frac{E}{4} &= \tan. \frac{E'}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E'}{4} &= \cot. \frac{E'}{4} \tan. \frac{E'}{4} \cot. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E''}{4} &= \cot. \frac{E'}{4} \cot. \frac{E'}{4} \tan. \frac{E''}{4} \cot. \frac{E'''}{4} \\ \cot.^2 \frac{E'''}{4} &= \cot. \frac{E'}{4} \cot. \frac{E'}{4} \cot. \frac{E''}{4} \tan. \frac{E'''}{4} \end{aligned} \right\} \dots (19).$$

By multiplying either of these sets, we should also obtain the reciprocal of equa. (17).

By means of (10) applied to both systems of associated triangles, we have

$$E + E' + E'' + E''' + E' + E' + E'' + E''' = 4\pi =$$

surface of the sphere . . . (20).

Taking the values of $\tan. \frac{E}{4} \tan. \frac{E'}{4}$ &c. from (1) and (16), we get

$$\left. \begin{aligned} \tan.^2 \frac{E}{4} \tan.^2 \frac{E'}{4} &= \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \\ \tan.^2 \frac{E'}{4} \tan.^2 \frac{E''}{4} &= \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \\ \tan.^2 \frac{E''}{4} \tan.^2 \frac{E'''}{4} &= \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \\ \tan.^2 \frac{E'''}{4} \tan.^2 \frac{E''}{4} &= \cot. \frac{s}{2} \tan. \frac{s-a}{2} \tan. \frac{s-b}{2} \tan. \frac{s-c}{2} \end{aligned} \right\} (21)$$

From the equality of the right sides of the last equations we find

$$\begin{aligned} \tan. \frac{E}{4} \tan. \frac{E'}{4} &= \tan. \frac{E'}{4} \tan. \frac{E''}{4} = \tan. \frac{E''}{4} \tan. \frac{E'''}{4} \\ &= \tan. \frac{E'''}{4} \tan. \frac{E''}{4} \dots (22). \end{aligned}$$

When the geographical positions of the three angles of a spherical

triangle are given to determine the area, we have the following expression, first given by the author of this Supplement, in the 12th volume of the Edinburgh Transactions, viz.

$$\frac{E}{\cos. \frac{1}{2}} = \left\{ \begin{array}{l} 1 + \cos. \alpha_1 \cos. \alpha_2 + \sin. \alpha_1 \sin. \alpha_2 \cos. \beta_1 - \beta_2 \\ + \cos. \alpha_1 \cos. \alpha_3 + \sin. \alpha_1 \sin. \alpha_3 \cos. \beta_1 - \beta_3 \\ + \cos. \alpha_2 \cos. \alpha_3 + \sin. \alpha_2 \sin. \alpha_3 \cos. \beta_2 - \beta_3 \\ 2(1 + \cos. \alpha_1 \cos. \alpha_2 + \sin. \alpha_1 \sin. \alpha_2 \cos. \beta_1 - \beta_2) \times \\ (1 + \cos. \alpha_1 \cos. \alpha_3 + \sin. \alpha_1 \sin. \alpha_3 \cos. \beta_1 - \beta_3) \times \\ (1 + \cos. \alpha_2 \cos. \alpha_3 + \sin. \alpha_2 \sin. \alpha_3 \cos. \beta_2 - \beta_3) \end{array} \right\}^{\frac{1}{2}} \dots (23);$$

where $\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3$, are the colatitudes and the longitudes of the vertices.

The spherical excess may also be very readily exhibited under another form by a direct investigation, but which, in my paper, in the Mathematical Repository, now publishing, is obtained by inference from another property. Thus,

$$\begin{aligned} \cot. \frac{E}{2} &= \frac{\sin. \frac{A+B+C-\pi}{2}}{\cos. \frac{A+B+C-\pi}{2}} = \frac{\sin. \frac{A+B+C}{2}}{\cos. \frac{A+B+C}{2}} \\ &= \frac{-\cos. \frac{A}{2} \cos. \frac{B}{2} \cos. \frac{C}{2} + \cos. \frac{A}{2} \sin. \frac{B}{2} \sin. \frac{C}{2} + \sin. \frac{A}{2} \cos. \frac{B}{2} \cos. \frac{C}{2} + \sin. \frac{A}{2} \sin. \frac{B}{2} \sin. \frac{C}{2}}{-\sin. \frac{A}{2} \sin. \frac{B}{2} \sin. \frac{C}{2} + \sin. \frac{A}{2} \cos. \frac{B}{2} \cos. \frac{C}{2} + \cos. \frac{A}{2} \sin. \frac{B}{2} \cos. \frac{C}{2} + \cos. \frac{A}{2} \cos. \frac{B}{2} \sin. \frac{C}{2}} \quad (24) \end{aligned}$$

Now, for these several functions of the angles, insert their values from (1, 2, page 77), keeping in mind that for the quantity there marked $\frac{1}{2}S$ (half sum), we have, in this supplement, written s ; and, performing the same upon each of the excesses, we shall obtain

$$\left. \begin{aligned} \cot. \frac{E}{2} &= \frac{-\operatorname{cosec}. s + \operatorname{cosec}. s - a + \operatorname{cosec}. s - b + \operatorname{cosec}. s - c}{-\sin. s + \sin. s - a + \sin. s - b + \sin. s - c} \left\{ \sin. s \sin. s - a \sin. s - b \sin. s - c \right\}^{\frac{1}{2}} \\ \cot. \frac{E'}{2} &= \frac{\operatorname{cosec}. s - \operatorname{cosec}. s - a + \operatorname{cosec}. s - b + \operatorname{cosec}. s - c}{\sin. s - \sin. s - a + \sin. s - b + \sin. s - c} \left\{ \sin. s \sin. s - a \sin. s - b \sin. s - c \right\}^{\frac{1}{2}} \\ \cot. \frac{E''}{2} &= \frac{\operatorname{cosec}. s + \operatorname{cosec}. s - a - \operatorname{cosec}. s - b + \operatorname{cosec}. s - c}{\sin. s + \sin. s - a - \sin. s - b + \sin. s - c} \left\{ \sin. s \sin. s - a \sin. s - b \sin. s - c \right\}^{\frac{1}{2}} \\ \cot. \frac{E'''}{2} &= \frac{\operatorname{cosec}. s + \operatorname{cosec}. s - a + \operatorname{cosec}. s - b + \operatorname{cosec}. s - c}{\sin. s + \sin. s - a + \sin. s - b + \sin. s - c} \left\{ \sin. s \sin. s - a \sin. s - b \sin. s - c \right\}^{\frac{1}{2}} \end{aligned} \right\} \dots (25).$$

Again, if for $s - a$, $s - b$, $s - c$, we insert their values in terms of r , r' , r'' , r''' , from (5, p. 225), we shall have the several areas of the associated triangles as functions of their inscribed radii.

$$\begin{aligned}
 \cot. \frac{E}{2} &= \frac{-\cot. r + \cot. r' + \cot. r'' + \cot. r'''}{-\tan. r + \tan. r' + \tan. r'' + \tan. r'''} \sqrt{\tan. r \tan. r' \tan. r'' \tan. r'''} \\
 \cot. \frac{E'}{2} &= \frac{\cot. r - \cot. r' + \cot. r'' + \cot. r'''}{\tan. r - \tan. r' + \tan. r'' + \tan. r'''} \sqrt{\tan. r \tan. r' \tan. r'' \tan. r'''} \\
 \cot. \frac{E''}{2} &= \frac{\cot. r + \cot. r' - \cot. r'' + \cot. r'''}{\tan. r + \tan. r' - \tan. r'' + \tan. r'''} \sqrt{\tan. r \tan. r' \tan. r'' \tan. r'''} \\
 \cot. \frac{E'''}{2} &= \frac{\cot. r + \cot. r' + \cot. r'' - \cot. r'''}{\tan. r + \tan. r' + \cot. r'' - \cot. r'''} \sqrt{\tan. r \tan. r' \tan. r'' \tan. r'''}
 \end{aligned}
 \quad \dots \dots (26).$$

We are also enabled, by means of the last set of equations, combined with (10), to ascertain the relation that subsists among the four radii r, r', r'', r''' , a relation which I believe has never before been assigned. For we have

$$\begin{aligned}
 \frac{E + E' + E'' + E'''}{2} &= \pi; \text{ and hence,} \\
 \tan. \frac{E + E' + E'' + E'''}{2} &= 0 = \frac{\Sigma_4(t) - \Sigma_4(t, t, t, t)}{1 - \Sigma_4(t, t) + \Sigma_4(t, t, t, t)}.
 \end{aligned}$$

Now this may be fulfilled in two different ways, either making the denominator infinite, whilst the numerator is finite; or making the numerator zero, whilst the denominator is a real quantity, finite or infinite. It would exceed the confined limits of a work like the present to discuss the circumstances of this function, and I shall, therefore, *assume* (though I shall elsewhere *prove* it) that the only condition that obtains is the latter. We hence have

* It is left for the student to prove, from the expression for the $\tan.$ of the sum of two arcs, at p. 48, that the expression for the $\tan.$ of four arcs is that in the text, in which $\Sigma(t)$ denotes the sum of the tangents of those arcs $\Sigma(t, t, t, t)$, the sum of their products taken two and two, and so on. The same may be generalized for any number of arcs.

$$\Sigma_4(t) - \Sigma_4(t, t, t) = 0;$$

which may be written as follows: (having previously divided it by

$$\tan. \frac{E}{2} \tan. \frac{E'}{2} \tan. \frac{E''}{2} \tan. \frac{E'''}{2}):$$

$$\left. \begin{aligned} & - \cot. \frac{E}{2} + \cot. \frac{E'}{2} \cot. \frac{E''}{2} \cot. \frac{E'''}{2} \\ & - \cot. \frac{E'}{2} + \cot. \frac{E}{2} \cot. \frac{E''}{2} \cot. \frac{E'''}{2} \\ & - \cot. \frac{E''}{2} + \cot. \frac{E}{2} \cot. \frac{E'}{2} \cot. \frac{E'''}{2} \\ & - \cot. \frac{E'''}{2} + \cot. \frac{E}{2} \cot. \frac{E'}{2} \cot. \frac{E''}{2} \end{aligned} \right\} = 0.$$

Insert for these cotangents their values, and reduce the expression to its simplest form. The work is somewhat laborious, but the result is comparatively simple; and hence I shall leave it as an exercise, for the student to perform alone. In another place I have given a different investigation of this formula, and several collateral topics are also combined with it, which will render it needless to enter into further detail upon this class of subjects, in the present necessarily very incomplete sketch. I trust, however, that enough is done to excite the interest of the mathematical student, whilst the extent of the subject itself will afford sufficient exercise for his ingenuity, and reward to his perseverance.

I am obliged to terminate these researches abruptly, on account of the space which they would occupy, if developed with any approach to completeness. I take the opportunity afforded me by reading the proofs, to state that my friend and neighbour, the Rev. Professor Logan, has also engaged in these and several collateral researches, and that the results to which both he and I may ultimately be found to have arrived, upon comparison of our mss., will be published in a single dissertation to be considered as our joint production. These researches will extend to every other function of parts of the spherical triangle, as well as those which have been in this supplement discussed; and to a considerable extension of each of these. It will then be seen that Spherical Geometry offers one of the most ample fields of research

that has yet been discovered; and I hope I shall not be thought too sanguine in anticipating that the properties of figures, traced upon the surface of the sphere, will, in a very few years, become as familiar to English Geometers as the correlative figures *in plano* now are.*

* Not only have Spherical Geometry and Spherical Trigonometry been greatly neglected in England, but also upon the Continent. The continental Geometers have, however, been truly assiduous in the cultivation of the Geometry of three divisions, and have imagined and discussed almost every variety of *method* for conducting their investigations in this branch of science: whilst, on the other hand, it will be difficult to point to any one British Geometer who ever added a single important theorem to our stock, much less devised a single original method of investigation. Of the *causes* of this humiliating fact, the present is not the place to speak. It may, however, be allowed me to mention what appears to be a barrier to our removing the discreditable charge. We have no work, expressly devoted to the subject, in which either the methods themselves are developed, or the spirit of them at all displayed. Mere *illustrations*, taken in a considerable degree at random from different works, in which they were originally very appropriately placed, when brought together without due regard to the principles themselves, and often without adapting the notation to any uniform standard—works like these, though they may be entitled treatises on the Geometry of Three Dimensions, can scarcely be called so without a complete perversion of the use of terms. He that renders a *method of investigation* intelligible, with whatever paucity of mere illustration, does more for the interests of science than he who collects all the illustrative examples of those methods that have ever been given into one single mass. Such collections are, indeed, too commonly calculated to confuse the young mind, and to repress all the ardour it might otherwise have felt.

Long ago, impressed with the importance of the subject, the author of this supplement formed the ambitious project of supplying this desideratum, and of furnishing a work in which the *spirit of the methods* which have been employed by the continental Geometers should be the first object of his anxiety. It has been his special aim, during the preparation of this work, to explain the essential character of each general principle, and to shew wherever they really differ from one another, and at the same time to illustrate each by a sufficient number of apposite examples, strictly adapted to the purpose for which they were employed. By proceeding thus with

every method that has been proposed by the different continental Geometers, and by furnishing also considerable portions of original results, it is hoped that a work may be produced which will render the study of solid Geometry scarcely more difficult than the more recondite portions of plane Geometry are now, and thereby greatly extend the cultivation of that branch of science in England.

He has been, however, led to think that a subsidiary elementary work on Descriptive Geometry would not be unacceptable to British Geometers, before the other goes to press. Even on this, the simplest of all the forms under which the Geometry of these dimensions presents itself, a merely graphic form—we have no treatise in England, nor yet a single chapter in any English course of Mathematics. There was indeed published in *America*, in 1821, a thin octavo, by M. Crozet, for the use of the Military College of the United States: but it would be scarcely less difficult to devise the methods originally than to acquire them from that treatise. Such a volume will therefore be sent to press with all convenient speed, the *avant courier* of the larger work.

NOTES.

NOTE (A). *page 53.*

We are to shew here that in the formula of De Moivre, viz.

$$\cos. \frac{n}{m} A \pm \sin. \frac{n}{m} A \cdot \sqrt{-1} = (\cos. A \pm \sin. A \cdot \sqrt{-1})^{\frac{n}{m}};$$

the first member has m values as well as the second. This fact we shall easily establish, by means of the property adverted to in the text, viz. that to any given values of the lines $\sin. A$, $\cos. A$, there correspond innumerable different arcs, viz. every arc in the infinite series,

$$A, 2\pi + A, 4\pi + A, 6\pi + A, \&c.$$

so that the first member of the above formula involves in it the following values, viz.

$$\cos. \frac{n}{m} A \pm \sin. \frac{n}{m} A \cdot \sqrt{-1}$$

$$\cos. \frac{n}{m} (2\pi + A) \pm \sin. \frac{n}{m} (2\pi + A) \cdot \sqrt{-1}$$

$$\cos. \frac{n}{m} (4\pi + A) \pm \sin. \frac{n}{m} (2\pi + A) \cdot \sqrt{-1}$$

$$\cos. \frac{n}{m} (6\pi + A) \pm \sin. \frac{n}{m} (6\pi + A) \cdot \sqrt{-1}$$

&c.

&c.

These values will continue different till we arrive at such a value, N , for one of the numeral coefficients, 1, 2, 4, 6, &c. as will render $\frac{n}{m} N\pi$ a multiple of 2π , when the first of the foregoing values will obviously

recur, so that by continuing the series we shall merely obtain a repetition of the former values. Now $\frac{n}{m} N\pi$ cannot become a multiple of 2π till N become equal to $2m$; hence we shall have expressed all the different values involved in the first member of De Moivre's formula when we have continued the above series of values as far as that in which the numeral coefficient is $2m - 2$; that is, when we have written m values. Hence each member of the formula involves m different values.

NOTE (B), page 108.

Professor Vince, at page 43 of his Trigonometry, has the following note.

“Difficulties have frequently arisen in consequence of its being supposed that an arc of 90° has a tangent and secant, each infinite. For instance, in a right-angled spherical triangle, radius : cosine of the angle at the base :: tangent of the hypotenuse : tangent of the base; now when the base = 90° , the hypotenuse = 90° ; and, therefore, these arcs being equal, if they have any tangents, of whatever value they may be, they must be equal; and, therefore, radius = cosine of the angle at the base, whatever that angle may be. This false conclusion arises from the supposition that an arc increases till it becomes 90° ; the tangent and secant increase without limit; and at 90° the arc ceases to have either a tangent or secant, by their definition. As the arc, by increasing, passes through 90° , the tangent and secant increase without limit, cease to exist at 90° , and then begin again at a quantity indefinitely great. And thus in other cases where the tangent or secant of an arc enter into the computation, when the arc becomes 90° , we can draw no conclusion on which we can depend.”

The foregoing reasoning is very much calculated to mislead the young student, although it does in reality tend to overturn the author's own hypothesis, and to show that the tangent of 90° must necessarily be infinite.

Taking the example chosen above, by Mr. Vince, we have for the true solution

$$\cos. \angle \text{ at base} = \text{rad. } \frac{\tan. 90^\circ}{\tan. 90^\circ}$$

which must necessarily involve the absurdity noticed above, except $\tan. 90^\circ$ be either 0 or ∞ ; but when the proper value ∞ is put for $\tan. 90^\circ$, then we have

$$\cos. \angle \text{ at base} = \text{rad. } \frac{\infty}{\infty} = \text{rad. } \frac{0}{0};$$

and as $\frac{0}{0}$ admits not only of the particular value 1, fixed upon by Mr.

Vince, but of an indefinite number of values, so does $\cos. \angle$ at base.

Upon the same grounds that Mr. Vince has rejected the tangent of 90° , he should have rejected the cosine of 90° , which, however, he admits to be 0.

For

$$\sin. \angle \text{ at base} = \text{rad. } \frac{\cos. \angle \text{ at vertex}}{\cos. \text{ base}};$$

but, when both base and hypotenuse are 90° , the angle at the vertex is 90° , and we ought therefore to have, according to Mr. Vince, $\sin. \angle$ at base = rad, which is, indeed, one solution, but by no means the only one, because the values of $\frac{0}{0}$ are innumerable.

NOTE (C), page 130.

It was shewn, at page 128, that if a ship in latitude x , vary her latitude by a very small portion Δx , and that she continue her course till her departure equals the difference of longitude due to the difference of latitude Δx , then the enlarged difference of latitude (Δy), due to this departure, will be

$$\Delta y = \sec. x \Delta x \therefore \frac{\Delta y}{\Delta x} = \sec. x.$$

This expression, it must be remembered, is nearer the truth the smaller we suppose Δx to be, and is, therefore, accurately true only when $\Delta x = 0$;

in other words, $\sec. x$ is the value to which the ratio $\frac{\Delta y}{\Delta x}$ continually approaches as we continually diminish Δx , (and in consequence Δy), and which value it actually becomes only when the terms of the ratio vanish, and the fraction takes the form $\frac{0}{0}$. By adopting the language of the *Differential Calculus* we have, in this case,

$$\frac{dy}{dx} = \sec. x \therefore dy = \sec. x dx = \frac{dx}{\cos. x}$$

$$\therefore y = \log. \tan. (45^\circ + \frac{1}{2} x), \text{ see } \textit{Int. Calculus}, \text{ p. 69};$$

the logarithm here used is the *Naperian*. To change it into a common logarithm we must multiply by the modulus 2.302585, &c.; it must be observed, however, that it is the logarithm of the *natural* tangent, which is here expressed, and not the tabular *logarithmic* tangent; it is, therefore, equal to the tabular logarithmic tangent minus 10. Hence, employing the table of logarithmic tangents, we may compute y from the formula

$$y = 2.302585 \{ \log. \tan. (45^\circ + \frac{1}{2} x) - 10 \} \times \text{Rad.}$$

and thus, as stated in the text, the meridional parts, y , corresponding to any given latitude x , may be expeditiously computed, independently of any previous computations.

The tables of meridional parts are usually expressed in nautical miles, and we shall have the number of miles in y , if, instead of multiplying by the radius of the earth, we multiply by the number of miles or minutes in it. Now in every circle the radius is equal to 3437.74679 minutes of that circle, because

$$3.14159, \&c. : 180^\circ :: 1 : 3437.74679 \text{ minutes};$$

hence, for the number of miles in y the expression is

$$7915.7044679 \{ \log. \tan. (45^\circ + \frac{1}{2} x) - 10 \};$$

or, since $\tan. 45^\circ + \frac{1}{2} x = \cot. 45^\circ - \frac{1}{2} x$, and, since, moreover,

$$\log. \cot. = \log. \frac{R^2}{\tan.} = 20 - \log. \tan.;$$

this expression may be written thus,

$$7915.7044679 \{ 10 - \log. \tan. (45^\circ - \frac{1}{2} x) \},$$

which gives the rule in the text.

We had intended to have introduced here some other particulars relating to Mr. Wright's projection of the meridian line, but we are precluded from doing so, as this treatise has already exceeded the limits assigned to it. We must, therefore, content ourselves with referring the student to Robertson's Navigation, vol. II. p. 135—146.

NOTE (D), page 220.

The following pretty theorems I have received from Mr. Lowry, since the first chapter on Spherical Geometry was in forms.

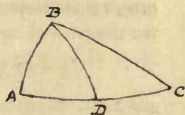
"Let ABC be a spherical triangle,* D the middle of one of the sides, AC; and let $\overline{AB} = d$. Then

$\overline{BD}$

$$\cos. a + \cos. c = 2 \cos. \frac{1}{2} b \cos. d.$$

For

$$\frac{\cos. a - \cos. \frac{1}{2} b \cos. d}{\sin. \frac{1}{2} b \sin. d} = \cos. BDC$$



$$\frac{\cos. c - \cos. \frac{1}{2} b \cos. d}{\sin. \frac{1}{2} b \sin. d} = \cos. BDA = -\cos. BDC.$$

Hence

$$\cos. a \cos. \frac{1}{2} b \cos. d = -\cos. c + \cos. \frac{1}{2} b \cos. d,$$

or

$$\cos. a + \cos. c = 2 \cos. \frac{1}{2} b \cos. d.$$

Cor. 1. When the triangle is inscribed in a ^{wh. may be a small $\odot$} semi-circle, the diameter of which is b ,

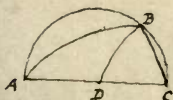
$$\cos. a + \cos. c = 2 \cos.^2 \frac{1}{2} b, \text{ or } \cos. a + \cos. c = 1 + \cos. b.$$

Cor. 2. And when $a = c$, we have $\cos. a = \cos.^2 \frac{b}{2}$.

Cor. 3. Hence, in a spherical square,† the cosine of the sides is equal to the square of the cosine of half the diagonal.

* The figure may be easily sketched by the student.

† A spherical four sided figure, whose sides are all equal, and whose angles are also all equal.



ABC the half of a small Circle.

Cor. 4. The sine of half the area of the triangle ACB in the circle is

$$= \tan. \frac{a}{2} \tan. \frac{c}{2}. \text{ Vide form. 20, Math. Repos. v. part 1. p. 7.}$$

Cor. 5. Hence, in a spherical rectangle,* the sine of one fourth of the area is equal to the rectangle of the semi-tangents of the two sides, that is,

$$= \tan. \frac{a}{2} \tan. \frac{c}{2}.$$

Cor. 6. And in the spherical square, the sine of $\frac{1}{2}$ area $= \tan.^2 \frac{a}{2}.$

Cor. 7. In a spherical parallelogram,† the sides of which are a, b, c, d , and diagonals h, h' , we shall have

$$\cos. a + \cos. b + \cos. c + \cos. d = 4 \cos. \frac{h}{2} \cos. \frac{h'}{2}.$$

These properties are very simple, but neat, and might serve as exercises in an elementary treatise.”

NOTE (E). page 235.

To account for some seeming discrepancies between the notes and text of this supplement, it is necessary to state that the *text* was drawn up in its present form from my manuscript, and the demonstration remodeled, (in many cases invented,) to adapt it to the isolated state of the portions here given, during brief intervals stolen from other pursuits and occupations, having but little alliance with these subjects. The *notes* were added afterwards, in a letter to Mr. Young, and distributed by him so as not to interfere (where the interference would occasion much change in the text already partly in *slips* and partly in forms,) with the part already in the

* A four sided spherical figure, all whose angles are equal; or, perhaps, better adapted to the term, we may call it the figure in which great circles bisecting the pairs of opposite sides intersect each other at right-angles.

† A figure whose opposite sides are equal.

These terms are adopted by analogy from plane Geometry. Perhaps it may be found desirable ere long to modify our terminology considerably: but it does not yet appear to be the time.

compositor's hands. Where addition could be worked into the text, and appeared more adapted to incorporation, it has been done; and where it did not coalesce with the text conveniently it has been put into foot notes. Some cases have, however, occurred where the addendum could not be properly made by either method, and it has therefore been altogether omitted. Still as these omissions are rather of an historical than mathematical nature, no inconvenience can result from them, except the possibly erroneous distribution of the names of discoverers of particular theorems. Should this be ultimately found to be the case, I trust the authors to whom they are erroneously attributed, as well as the authors to whom they are actually due, will excuse the undesigned mistake.

There is, however, one particular case to which I wish more especially to refer, since I had till just now considered a theorem upon which I set some value, (and which, indeed, was the origin of my researches upon these topics,) to be original, when, in fact, it had been discovered more than a quarter of a century ago, by Professor Lowry. I have just received a note from that distinguished Geometer, containing, amongst other matters, a reference to the *Mathematical Repository*, N.S. vol. I. p. 157.

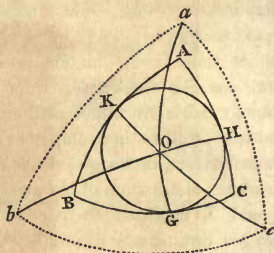
Upon turning to this volume I find an anticipation of this beautiful property of the polar triangles: but, as my copy of the *Repository* had been lent to a friend during the whole of the time I had the subject before my mind, my own discovery was perfectly independent of his, though so long posterior to it. I am quite sure, indeed, that I had never read that passage, or so beautiful a property must have been inevitably laid up amongst my collections. I am happy, however, to be able to render back to Professor Lowry the credit of the priority of discovery in the same volume in which I had seemed to claim not only independence but priority.

His demonstration (as was to be expected when the methods of spherical research in general at the two periods are compared) differs totally from mine: but his, as the geometrical often will have over the analytical, even when the latter is cultivated to its utmost degree of perfection, has advantages over mine, which renders it desirable to give it here. It is simple, and it proves more than mine proves, or perhaps can prove in moderate compass, viz. that the centres of the two circles, whose radii are complementary, coalesce with one another. I will add, that *to him alone we owe every important spherical theorem that can be set down to the credit of Englishmen during at least a century past, probably even longer.*

Find the centre O of the inscribed circle in ABC , and from the points of contact G, H, K , draw the radii OG, OH, OK . Then these being perpendicular to the sides BC, CA, AB , respectively pass through the poles a, b, c , of those sides. Hence, by polar triangles,

$$aOG = bOH = cOK. = \frac{\pi}{2}. \quad \text{But}$$

$OG = OH = OK$, and, therefore, $Oa = Ob = Oc$, or O is also the centre of the circle about the polar triangle abc : that is, the centres of the primary inscribed and the polar circumscribed circles are coincident. In the same manner the centres of polar inscribed and primary circumscribed are coincident. And it has been shewn that these radii are complementary.



I may remark that the expression for the distance of the inscribed and circumscribed centres, *in terms of the radii themselves*, has not yet been given. *In plano* that distance was so assigned by *Mr. Landen*, and has been very elegantly investigated by *Mr. Lowry*, in the *Mathematical Repository*. The corresponding problem has been several times attempted, but other parts of the triangle have appeared in every result that has yet been published. The neatest form that I have seen is given anonymously in the *Annales des Mathematiques*, tom. vi. p. 223, viz.

$$\cos. D = \frac{\sin. a + \sin. b + \sin. c}{\sqrt{\sin. s \sin. s - a \sin. s - b \sin. s - c}} \cdot \sin. r \cos. R.$$

THE END.

MatTr
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J. B. Good

Young

Author

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